Stability of LTI systems

$$\dot{x}(t) = Ax(t)$$

Asymptotic stability means that $\|x(t)\| \rightarrow 0$ as t goes to infinity, for all initial conditions $x(0)$. Let $p(s)=\det(sI-A)$ the characteristic polynomial of A.

1) The system is asymptotically stable if and only if all the n eigenvalues of matrix A are in the open left hand plane.
2) The system is asymptotically stable if and only if the variation of the phase of $p(j\omega)$ is $n\pi/2$, as ω varies from 0 to infinity (Michaelov criterion).
3) The system is asymptotically stable if and only if there exists $P=P'>0$ satisfying $A'P+PA<0$ (Lyapunov Lemma).

The family of interval polynomials

$$p(s) = \sum_{i=1}^{n} \left[a_i^-, a_i^+\right] s^i$$

with independent coefficients in the given intervals is robustly Hurwitz if and only if the following four Kharitonov polynomials are Hurwitz.

$$p_1(s) = a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + a_4^- s^4 + \cdots$$
$$p_2(s) = a_0^- + a_1^+ s + a_2^+ s^2 + a_3^- s^3 + a_4^- s^4 + \cdots$$
$$p_3(s) = a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + a_4^+ s^4 + \cdots$$
$$p_4(s) = a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + a_4^+ s^4 + \cdots$$

Adjoin system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ G(s) &= C(sI-A)^{-1}B + D \end{aligned} \quad \longrightarrow \quad \begin{aligned} \dot{\lambda} &= -A'\lambda - C'\bar{u} \\ \bar{y} &= B'\lambda + D'\bar{u} \\ G(-s)' &= B'(-sI-A')^{-1}C' + D' \end{aligned}$$

Space $L_2(\tau,T)$

This space is defined by all (real valued, matrix) functions of time defined in (τ,t) and zero elsewhere such that

$$\int_{\tau}^{T} trace\left[v(t)'v(t)\right]d\omega < \infty$$

Very important in engineering applications are the spaces $L_2(-\infty,0)$ and $L_2(0,\infty)$. Take for instance the last space.
The norm is of

$$v \in L_2(0,\infty) \qquad \|v\|_2 = \left(\int_0^{\infty} trace\left[v(t)'v(t)\right]dt\right)^{1/2}$$

The L_2 space

L_2 space: The set of (rational) functions G(s) such that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} trace\left[G(-j\omega)'G(j\omega)\right]d\omega < \infty$$

(strictly proper – no poles on the imaginary axis!)

With the inner product of two functions G(s) and F(s):

$$\langle G(s),F(s)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} trace\left[G(-j\omega)'F(j\omega)\right]d\omega$$

The space L_2 is a pre-Hilbert space. Since it is also complete, it is indeed a Hilbert space. The norm, induced by the inner product, of a function G(s) is

$$\|G(s)\|_2 = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} trace\left[G(-j\omega)'G(j\omega)\right]d\omega\right]^{1/2}$$

The subspaces H_2 and $H_2^{\perp}$

The subspace H_2 is constituted by the functions of L_2 which are analytic in the right half plane. (strictly proper – stable !) . The subspace $H_2^{\perp}$ is constituted by the functions of L_2 which are analytic in the left half plane. (stricty proper – antistable !)

Note: A rational function in L_2 is a strictly proper function without poles on the imaginary axis. A rational function in H_2 is a strictly proper function without poles in the closed right half plane. A rational function in $H_2^{\perp}$ is a strictly proper function without poles in the closed left half plane. The functions in H_2 are related to the square summable functions of the real variable t in $(0,\infty]$. The functions in $H_2^{\perp}$ are related to the square summable functions of the real variable t in $(-\infty,0]$.

A function in L_2 can be written in an unique way as the sum of a function in H_2 and a function in $H_2^{\perp}$: $G(s)=G_1(s)+G_2(s)$. Of course, $G_1(s)$ and $G_2(s)$ are orthogonal, i.e. $<G_1(s),G_2(s)>=0$, so that

$$\|G(s)\|_2^2 = \|G_1(s)\|_2^2 + \|G_2(s)\|_2^2$$

System theoretic interpretation of the H_2 norm

Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) \\ x(0) &= 0 \end{aligned}$$

and let G(s) be its transfer function. Also consider the quantity to be evaluated:

$$J_1 = \sum_{i=1}^{m} \int_0^{\infty} z^{(i)}{}'(t) z^{(i)}(t)dt$$

where $z^{(i)}$ represents the output of the system forced by an impulse input at the i-th component of the input.

$$J_1 = \sum_{i=1}^{m}\int_0^{\infty} z^{(i)}{}'(t)z^{(i)}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{i=1}^{m} Z^{(i)}(-j\omega)'Z^{(i)}(j\omega)d\omega =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{i=1}^{m} e_i'G(-j\omega)'G(j\omega)e_i d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} trace[G(-j\omega)'G(j\omega)] = \|G(s)\|_2^2$$

Computation of the norm
Lyapunov equations

"Control-type Lyapunov equation"

$$A'P_o + P_o A + C'C = 0$$

"Filter-type Lyapunov equation"

$$P_r A' + P_r A + BB' = 0$$

$$\|G(s)\|_2^2 = J_1 = \sum_{i=1}^{m} \int_0^\infty z^{(i)}{}'(t) z^{(i)}(t)dt = \int_0^\infty \sum_{i=1}^{m} trace\left[e_i' B' e^{A't} C'Ce^{At} Be_i\right] dt =$$

$$= \int_0^\infty trace\left[e^{A't} C'Ce^{At} \sum_{i=1}^{m} Be_i e_i' B' \right] dt$$

$$= trace\left[B' \int_0^\infty e^{A't} C'Ce^{At} dt \; B \right] = trace\left[B' P_0 B\right]$$

$$= trace\left[C \int_0^\infty e^{At} BB' e^{A't} dt \; C' \right] = trace\left[CP_r C'\right]$$

Other interpretations

1. Assume now that w is a white noise with identity intensity and consider the quantity:

$$J_2 = \lim_{t\to\infty} E(z'(t)z(t))$$

It follows:

$$J_2 = \lim_{t\to\infty} trace\left[E\left[\int_0^t Ce^{A(t-\tau)} Bw(\tau)d\tau \int_0^t w(\sigma)'B'e^{A'(t-\sigma)} C' d\sigma \right]\right] E =$$

$$= \lim_{t\to\infty} trace\left[\int_0^t \int_0^t Ce^{A(t-\tau)} BE\left[w(\tau)w(\sigma)'\right] B' e^{A'(t-\sigma)} C' d\tau d\sigma \right] =$$

$$= \lim_{t\to\infty} trace\left[\int_0^t Ce^{A(t-\tau)} BB' e^{A'(t-\tau)} C' d\tau \right] = \|G(s)\|_2^2$$

Finally, consider the quantity

$$J_3 = \lim_{T\to\infty} \frac{1}{T} E\int_0^T z'(t)z(t)dt$$

and let again w(.) be a white noise with identity intensity. It follows:

$$E(z'(t)z(t)) = trace[CP(t)C'], \quad P(t) = \int_0^t e^{A\tau}BB'e^{A'\tau}d\tau$$

Notice that $J_3 = (\|G(s)\|_2)^{1/2}$ since

$$\dot{P}(t) = AP(t) + P(t)A' + BB', \quad AY + YA' + BB' = 0, \quad Y = \lim_{T\to\infty} \frac{1}{T}\int_0^T P(\tau)d\tau$$

2. Consider again the space $L_2(0,\infty)$ of time signals v(t), i.e. such that

$$\int_0^\infty v(t)'v(t)dt < \infty$$

Hence, for $G(s)=C(sI-A)^{-1}B \in H_2$ we have

$$\sup_{u\in L_2(0,\infty)} \frac{\sup_{t>0} y(t)'y(t)}{\int_0^\infty u(t)'u(t)dt} = \|C'SC\|$$

where $S>0$ is the unique solution of $AS+SA'+BB'=0$.

This means that (in the SISO case) the H_2 norm corresponds to the worst peak value of the output when the input is a bounded energy signal.

Remark

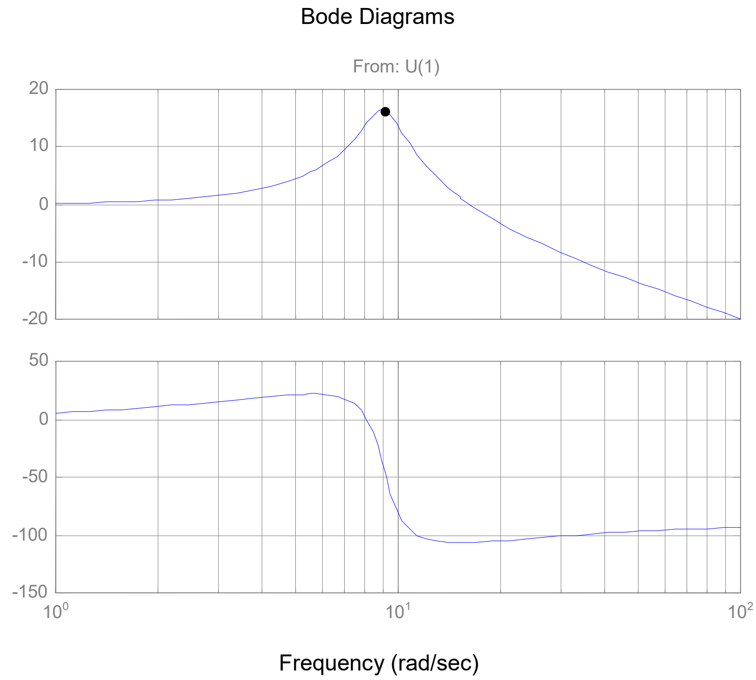
- $\|G(s)\|_2^2 = \sum_i \sum_j \|G_{ij}(s)\|_2^2$
- $a \neq 0, \quad \left\| \frac{\kappa}{s+a} \right\|_2^2 = \frac{\kappa^2}{2|a|}$
- $a \neq 0, b > 0 \quad \left\| \frac{\kappa s + 1}{s^2 + as + b} \right\|_2^2 = \frac{1+\kappa^2 b}{2b|a|}$

Example

Compute (exploit the definition and the Pythagoras theorem) the L_2 norm of

$$G(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{2}{s+2} & 0 \\ \frac{1}{s+10} & 0 & \frac{s+1}{s^2-s+5} \end{bmatrix}$$

What the L_∞ norm is?



Given a SISO system with transfer function G(s), the L_∞ space is the space of all G(s) such that

$$\sup_\omega |G(j\omega)| < \infty$$

For this, it is necessary and sufficient that the (rational) function G(s) is proper with no poles on the imaginary axis. The space H_∞ is composed by proper (rational) functions G(s) with all poles with strictly negative real parts. In both cases the norm is given by

$$\|G(s)\|_\infty = \sup_\omega |G(j\omega)|$$

In the multivariable case

$$\|G(s)\|_\infty = \sup_\omega \|G(j\omega)\|$$

where $\|\Omega\| = \overline{\sigma}(\Omega) = \sqrt{\lambda_{\max}(\Omega^*\Omega)} = \sqrt{\lambda_{\max}(\Omega\Omega^*)}$

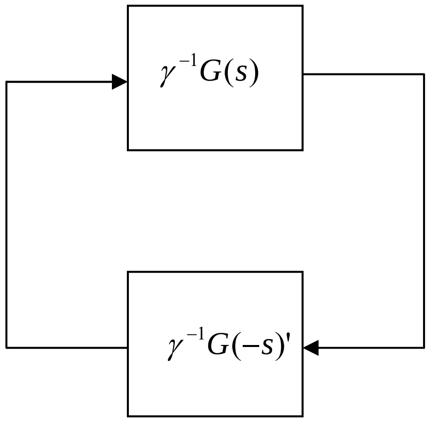
$$\textit{If } G(s) \textit{ is rational stable proper } (H_\infty)$$

$$\|G(s)\|_\infty = \sup_{\mathrm{Re}(s)\geq 0} \|G(s)\| = \sup_\omega \|G(j\omega)\|$$

For the frequency-domain definition we can consider a transfer function without poles on the imaginary axis. It is easy to understand that the norm of a function in L_∞ can be recast to the norm of its stable part given through the so-called inner-outer factorization. As an example,

$$\left\|\frac{s+5}{(s-2)(s+3)}\right\|_\infty = \left\|\frac{s+5}{(s-2)(s+3)}\frac{s-2}{s+2}\right\|_\infty = \left\|\frac{s+5}{(s+2)(s+3)}\right\|_\infty$$

The Hamiltonian matrix



Assume that $G(s)=C(sI-A)^{-1}B+D$ in minimal form, and let γ be a positive number. Then,

Theorem

$$\|G(s)\|_\infty < \gamma \quad \Longleftrightarrow \quad \begin{cases} \det(I - G(-j\omega)'G(j\omega)) \neq 0, \ \omega \in [0,\infty) \\ \|G(j\infty)\| < \gamma \end{cases}$$

Proof: This is equivalent to say that $\|D\|<\gamma$ and the eigenvalues of the closed-loop system in the figure do not lie on the imaginary axis. But it can be easily verified that the eigenvalues are those of the Hamiltonian matrix

$$H = \begin{bmatrix} A + B(\gamma^2 I - D'D)^{-1}D'C & B(\gamma^2 I - D'D)^{-1}B' \\ -C'(I - \gamma^{-2}DD')^{-1}C & -A' - C'D(\gamma^2 I - D'D)^{-1}B' \end{bmatrix}$$

Time-domain characterization

$G(s) = D+C(sI-A)^{-1}B$

$\dot{x} = Ax + Bw$

$z = Cx + Dw$

A = asymptotically stable

$$\|G(s)\|_\infty = \sup_{w\in L_2} \frac{\|z\|_2}{\|w\|_2}$$

There does not exist a direct procedure to compute the infinity norm. However, it is easy to establish whether the norm is bounded from above by a given positive number γ.

The symbol L_2 indifferently the space of square integrable or the space of the strictly proper rational function without poles on the imaginary axis. If w(.) is a white noise, the infinity norm represents the square root of the maximal intensity of the output spectrum.

BOUNDED REAL LEMMA

Let γ be a positive number and let A be asymptotically stable.

Theorem

The three following conditions are equivalent each other:

(i) $\|G(s)\|_\infty < \gamma$

(ii) $\|D\|<\gamma$ and there exists the positive semidefinite stabilizing solution of the Riccati equation

$$A'P + PA + (PB + C'D)\left(\gamma^2 I - D'D\right)^{-1}(B'P + DC') + C'C = 0$$

$$A + B\left(\gamma^2 I - D'D\right)^{-1}(B'P + DC') \quad stable$$

(iii) $\|D\|<\gamma$ and there exists a positive definite solution of the Riccati inequality

$$A'P + PA + (PB + C'D)\left(\gamma^2 I - D'D\right)^{-1}(B'P + DC') + C'C < 0$$

Comments

Notice that as γ tends to infinity, the Riccati equation becomes a Lyapunov equation that admits a (unique) positive semidefinite solution, thanks to the system stability. This is obvious if one notices that the infinity norm of a stable system is finite.

Proof of Theorem

For simplicity, let consider the case where the system is strictly proper, i.e. D=0. The equation and the inequality are

$$A'P + PA + \frac{PBB'P}{\gamma^2} + C'C = (<)0$$

Also denote: $G(s)^{\sim}=G(-s)'$.

Points (ii)→(i) and (iii) →(i).

Assume that there exists a positive semidefinite (definite) solution P of the equation (inequality). We can write:

$$(sI + A')P - P(sI - A) + \frac{PBB'P}{\gamma^2} + C'C = (<)0$$

Premultiply to the left by $B'(sI+A')^{-1}$ e to the right by $(sI-A)^{-1}B$, it follows (spectral factorization)

$$G^{\sim}(s)G(s) = \gamma^2 I - T^{\sim}(s)T(s)$$
$$T(s) = \gamma I - \gamma^{-1}B'P(sI - A)^{-1}B$$

so that $\|G(s)\|_\infty < \gamma$.

Points (i)→(ii)

Assume that $\|G(s)\|_\infty < \gamma$. We now proof that the Hamiltonian matrix

$$H = \begin{bmatrix} A & \gamma^{-2}BB' \\ -C'C & -A' \end{bmatrix}$$

does not have eigenvalues on the imaginary axis. Indeed, if, by contradiction $j\omega$ is one such eigenvalue, then

$$(j\omega - A)x - \gamma^{-2}BB'y = 0$$
$$(j\omega + A')y + C'Cx = 0$$

Hence $Cx = -\gamma^{-2}C(j\omega I-A)^{-1}BB'(j\omega I+A)^{-1}C'Cx$, so that $G(-j\omega)'G(j\omega)Cx=0$ and $Cx=0$. Consequently, y=0 and x=0, that is a contradiction. Then, since the Hamiltonian matrix does not have imaginary eigenvalues, it must have 2n eigenvalues, n of them having negative real parts. The remaining on eigenvalues are the complex conjugate of the previous ones. This fact follows from the matrix being Hamiltonian, i.e. satisfying $JH+H'J=0$, where $J=[0\ I;-I\ 0]$. Let take the n-dimensional subspace generated by the (generalized) eigenvectors associated with the stable eigenvalues and let choice a matrix $S=[X^*\ Y^*]^*$ whose range coincides with such a subspace. For a certain asymptotically stable matrix T (restriction of H) it follows $HS=ST$. We now proof that $X^*Y=Y^*X$. Indeed let define $V=X^*Y-Y^*X=S^*JS$ and notice that $VT=S^*JST=S^*JHS=-S^*H'JS=-T^*S^*JS=-T^*V$, so that $VT+T^*V=0$. The stability of T yields (by the well known Lyapunov Lemma) that the unique

solution is V=0, i.e. X*Y=Y*X. We now proof that X is invertible. Indeed from HS=ST it follows that $AX+\gamma^{-2}BB'Y=XT$ and $-A'Y-C'CX=YT$. Premultiplying the first equation by Y* yields $Y^*AX+\gamma^{-2}Y^*BB'Y=Y^*XT=X^*YT$. Hence if, by contradiction, $v\in Ker(X)$ then $v^*Y^*BB'Yx=0$ so that $B'Yv=0$. From the first equation we have XTv=0 and $-A'Yv=YTv$. In conclusion, if $v\in Ker(X)$ then $Tv\in Ker(X)$. By induction it follows $T^kv\in Ker(X)$, and again
$-A'YT^kv=YT^{k+1}v$, k nonnegative. Take the monic polynomial of minimum degree h(T) such that h(T)v=0 (notice that it always exists) and write $h(T)=(\lambda I-T)m(T)$. Since T è asymptotically stable, it results $Re(\lambda)<0$. Obviously $q=m(T)v\neq 0$. Then $q\in Ker(X)$, $Tq=\lambda q$, $-A'Yq=YTq=\lambda Yq$. Since A is asymptotically stable and $Re(\lambda)<0$, this last equation implies Yq=0, that, together with Xq=0 implies q=0, thanks to the n-dimensionality of the range of S. This is a contradiction. So we have proven that X is invertible. Hence, defining $P=YX^{-1}$ and noticing that P*=P one has $AX+\gamma^{-2}BB'Y=XT$ and $-A'Y-C'CX=YT$ so that $-A'Y-C'CX=YX^{-1}(AX+\gamma^{-2}BB'Y)$ and $A'P+C'C+PA+\gamma^{-2}PBB'P=0$.

Besides being hermitian, P is also real (and therefore symmetric). Indeed, we can write $[X_c^*\ Y_c^*]^*N=[X^*\ Y^*]$, where N is a permutation matrix and X_c, Y_c are complex matrices which are the complex conjugates of X e Y, respectively. Hence, if Pc is the complex conjugate of P, one has

$P=YX^{-1}=Y_cNN^{-1}X_c^{-1}=Y_cX_c^{-1}=P_c$.

Finally, the fact that P is positive semidefinite is again a consequence of the Lyapunov Lemma, applied to the Riccati equation.

Points (i)→(iii)

Assume that $\|G(s)\|_\infty<\gamma$ and define

$$\overline{G}(s)=\begin{bmatrix} C \\ \sqrt{\varepsilon}I \end{bmatrix}(sI-A)^{-1}B,\quad 0<\varepsilon<\frac{\gamma^2-\|G(s)\|_\infty^2}{\|(sI-A)^{-1}B\|_\infty^2}$$

Then,

$$\overline{G}^{\sim}(s)\overline{G}(s)=G^{\sim}(s)G(s)+\varepsilon F^{\sim}(s)F(s),\quad F(s)=(sI-A)^{-1}B$$

and hence

$$\|\overline{G}(s)\|_\infty^2\le\|G(s)\|_\infty^2+\varepsilon\|F(s)\|_\infty^2<\gamma$$

Then, from the implication (i)→(ii) it follows that there exists the positive semidefinite and stabilizing solution of the Riccati equation

$A'P+PA+\gamma^{-2}PBB'P+C'C+\varepsilon I=0$, so that P>0 and P solves $A'P+PA+\gamma^{-2}PBB'P+C'C<0$.

Worst Case

The "worst case" interpretation of the H_∞ norm is given by the following result:

Theorem

Let A be Hurwitz, $\|G(s)\|_\infty < \gamma$ and let x_0 be the initial state of the system. Then,

$$\sup_{w\in L_2} \|z\|_2^2 - \gamma^2\|w\|_2^2 = x_0' P x_0$$

where P is the solution of the BRL Riccati equation.

Proof of Theorem

Consider the function V(x)=x′Px and its derivative along the trajectories of the system. Letting $\Delta=(\gamma^2 I-D'D)^{-1}$ we have:

$$\begin{aligned}\dot V &= x'(A'P+PA+PBw+B'PB)x = -x'C'Cx - x'(PB+C'D)\Delta(B'P+D'C)x = -z'z \\ &\quad + w'(D'C+B'P)x + x'(C'D+PB)w + w'D'Dx - x'(PB+C'D)\Delta(B'P+D'C)x = \\ &= -z'z + \gamma^2 w'w - (w-w_{ws})'\Delta^{-1}(w-w_{ws})\end{aligned}$$

where $w_{ws} = \Delta(B'P+D'C)x$ *is the worst disturbance*. Recalling that the system is asymptotically stable and taking the integral of both hands, the conclusion follows.

Observation: LMI

Schur Lemma

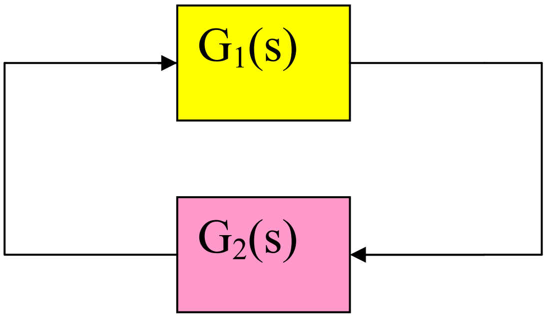
$$P > 0$$

$$A'P + PA + (PB + C'D)\left(\gamma^2 I - D'D\right)^{-1}(B'P + DC') + C'C < 0$$

$$\Updownarrow$$

$$\begin{bmatrix} -A'P - PA - C'C & PB + C'D \\ B'P + D'C & \gamma^2 I - D'D \end{bmatrix} > 0$$

The Small Gain Theorem



Theorem

Assume that $G_1(s)$ is stable. Then:

(i) The interconnected system is stable for each stable $G_2(s)$ with $\|G_2(s)\|_\infty<\alpha$ if $\|G_1(s)\|_\infty \le \alpha^{-1}$.

(ii) If $\|G_1(s)\|_\infty>\alpha^{-1}$ then there exists a stable $G_2(s)$ with $\|G_2(s)\|_\infty<\alpha$ that destabilizes the interconnected system.

Proof of Theorem (in the scalar case the proof easily follows from the Nyquist criterion)

Point (i).

If $\|G_2(s)\|_\infty<\alpha$ and $\|G_1(s)\|_\infty\le\alpha^{-1}$, then $\det[I-G_1(s)G_2(s)]\neq 0$, for $Re(s)\ge 0$. This fact, together with the stability of $G_1(s)$ and $G_1(s)$, is equivalent to the stability of the closed-loop system (the simple check is left to the reader).

Point (ii).

For the proof of this theorem, let consider the case where the number m of column of $G_1(s)$ is less than or equal to the number p of columns of $G_2(s)$. The proof in the converse case is similar. Then, assume that $\|G_1(s)\|_\infty=\alpha^{-1}(1+\varepsilon)=\rho^{-1}$, $\varepsilon>0$, and write the singular value decomposition of $G_1(j\omega)$, i.e. $G_1(j\omega)=U(j\omega)\Sigma(j\omega)V^{\sim}(j\omega)$ where $\Sigma(j\omega)=[S(j\omega)'\ 0]'$ and $S(j\omega)$ is square with dimension m. Moreover, take a stable $G_2(s)$ such that $G_2(j\omega)=\rho V(j\omega)[I\ \ 0]U^{\sim}(j\omega)$. Notice that $G_2^{\sim}(j\omega)G_2(j\omega)=\rho^2<\alpha^2$ so that $\|G_2(s)\|_\infty<\alpha$. We have:

Since $\|G_1(s)\|_\infty=\rho^{-1}$, it follows that there exists a frequency b such that $\lim_{\omega\to b}\sigma(G_1(\omega))=\rho^{-1}$. Hence, being $S(j\omega)$ diagonal with the singular values of $G_1(j\omega)$ on the diagonal, it follows that at least one entry of $\rho S(j\omega)$ goes to zero as ω tends to b. In conclusion $\lim_{\omega\to b}\det[I-\rho S(j\omega)]=0$, so that the closed-loop system is not stable.

$$\det[I-G_1(j\omega)G(j\omega)]=\det\left[I-\rho U(j\omega)\begin{bmatrix}S(j\omega) & 0\\ 0 & 0\end{bmatrix}U^{\sim}(j\omega)\right]$$

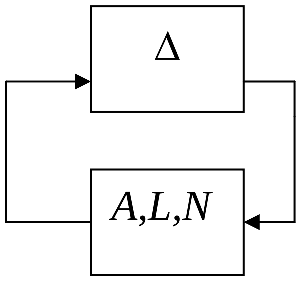
$$=\det\left[\left[U(j\omega)U^{\sim}(j\omega)-\rho U(j\omega)\begin{bmatrix}S(j\omega) & 0\\ 0 & 0\end{bmatrix}U^{\sim}(j\omega)\right]\right]$$

$$=\det\left[\begin{bmatrix}I-\rho S(j\omega) & 0\\ 0 & I\end{bmatrix}\right]=\det[I-\rho S(j\omega]$$

H_∞ and quadratic stability

$\dot{x} = (A + L\Delta N)x$, $\|\Delta\| \le \alpha$

$$\begin{cases} \dot{x} &= Ax + Lw \\ z &= Nx \\ w &= \Delta x \end{cases}$$



The system is said to be quadratically stable if there exists a solution (Lyapunov function) to the following inequality, for every Δ in the set $\|\Delta\| \le \alpha$,

$$(A + L\Delta N)^* P + P(A + L\Delta N) < 0$$

Theorem

The system is quadratically stable if and only if

$\|N(sI-A)^{-1}L\|_\infty < \alpha^{-1}$.

Time-varying, complex

Proof of Theorem

First observe that the following inequality holds:

$$
\begin{aligned}
&(A+L\Delta N)'P+P(A+L\Delta N)= \\
&A'P+PA+N'\Delta'\Delta N+PLL'P-(N'\Delta'-PL)(\Delta N-L'P) \\
&\leq A'P+PA+PLL'P+N'N\alpha^2=\alpha^2(A'X+XA+\frac{XLL'X}{\alpha^{-2}}+N'N)
\end{aligned}
$$

where $P\alpha^2=X$. Hence, if there exists X>0 satisfying

then A+LΔN is asymptotically stable for every Δ, $||\Delta|| \leq \alpha$, with the same Lyapunov function (quadratic stability). This

$$A'X+XA+\frac{XLL'X}{\alpha^{-2}}+N'N<0$$

happens if $||N(sI-A)^{-1}L||_\infty<\alpha^{-1}$. In conclusion, we have proven that the condition $||N(sI-A)^{-1}L||_\infty<\alpha^{-1}$ implies that the system is quadratically stable. Vice-versa assume that that the system is quadratically stable. In particular the system is robustly stable, i.e. stable for each Δ in the set $||\Delta||\leq\alpha$. Hence, for each $||\Delta||\leq\alpha$ it results

$$(*) \quad \det\left[I-\Delta'G(-s)'G(s)\Delta\right]\neq 0, \quad \mathrm{Re}(s)\geq 0$$

Assume by contradiction that $||G(s)||_\infty\geq\alpha^{-1}$, i.e. there exists b such that

$$\lambda_{\max}\left(I-\alpha G(-jb)'G(jb)\alpha\right)\leq 0$$

Since $\lambda_{\max}(I-\alpha G(-\infty)'G(\infty)\alpha)=1>0$, we have that there exists $s=j\omega$ that violates (*), a contradiction.

Entropy

Consider a stable system G(s) with state space realization (A,B,C,0), and assume that $\|G(s)\|_\infty < \gamma$.

The γ-entropy of the system is defined as

$$I_\gamma(G) = -\frac{\gamma^2}{2\pi}\int_{-\infty}^{+\infty} \ln \det\left[I - \frac{G(-j\omega)'G(j\omega)}{\gamma^2}\right] d\omega$$

Proposition

$$\|G(s)\|_2^2 \le I_\gamma(G) \le \left(\frac{-\log(1-\alpha^2)}{\alpha^2} \right) \|G(s)\|_2^2 \,, \quad \alpha = \frac{\|G(s)\|_\infty}{\gamma}$$

$$I_\gamma(G) = trace\,[B'PB] = trace\,[CQC']$$

Where P and Q are the stabilizing solutions of the Riccati equations

$$A'P + PA + \gamma^{-2}PBB'P + C'C = 0$$
$$AQ + QA' + \gamma^{-2}QC'CQ + BB' = 0$$

Comments

It is easy to check that the γ-entropy measure is not a norm, but can be considered as a generalization of the square of the H_2 norm

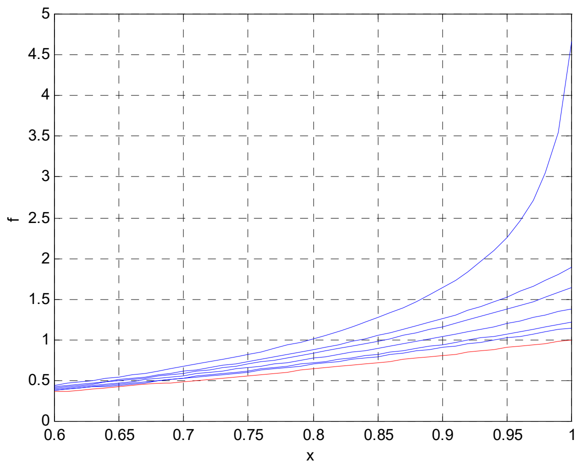
$$\|G(s)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sum_{i=1}^{m} \sigma_i^2 [G(j\omega)]\, d\omega$$

Indeed the γ-entropy can be written as

$$I_\gamma(G) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{i=1}^{m} f(\sigma_i^2[G(j\omega)])\, d\omega$$

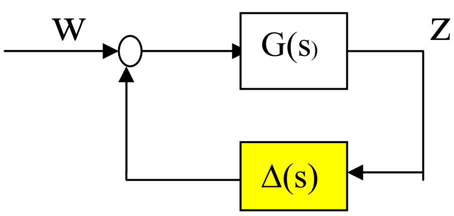
$$f(x^2) = -\gamma^2 \ln(1-\frac{x^2}{\gamma^2})$$

Graph of $f(x^2)$ parametrized in $\gamma>1$. For large γ the function $f(x^2)$ gets closer to x^2 (red line).



Comments

An interpretation for the γ-entropy for SISO systems is as follows. Consider the feedback configuration:



where w(.) is a white noise and Δ(s) is a random transfer function with $\Delta(j\omega_1)$ independent on $\Delta(j\omega_2)$ and uniformly distributed on the disk of radius γ^{-1} in the complex plane.

Hence the expectation value over the random feedback transfer function is

$$\underset{\Delta}{E}\left(\left\|\frac{G(s)}{1-G(s)\Delta(s)}\right\|_2^2\right) = I_\gamma(G)$$

Proof of the Proposition

First notice that $f(x^2) \geq x^2$ so that the conclusion that $I_\gamma(G) \geq \|G(s)\|^2_2$ follows immediately.
Now, let

$$\beta = \frac{1}{\alpha^2} = \frac{\gamma^2}{\|G(s)\|_\infty^2}, \quad r_i = \frac{\gamma^2}{\sigma_i^2(G(j\omega))}$$

Of course it is $r_i \geq \beta > 1$. Then

$$\left(1-\frac{1}{r_i}\right)^{r_i} \geq \left(1-\frac{1}{\beta}\right)^{\beta} \longrightarrow \ln\left(1-\frac{1}{r_i}\right) \geq \frac{\beta}{r_i}\ln\left(1-\frac{1}{\beta}\right)$$

so that

$$I_\gamma(G) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{i=1}^{m} -\gamma^2 \ln\left(1-\frac{1}{r_i}\right)d\omega \leq -\beta\ln\left(1-\frac{1}{\beta}\right)\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{i=1}^{m}\frac{\gamma^2}{r_i}d\omega$$

which is the conclusion. In order to prove that the entropy can be computed from the stabilizing solution of the Riccati equation, recall (Theorem 1) that, since $\|G(s)\|_\infty < \gamma$, it is possible to write

$$G^{\sim}(s)G(s) = \gamma^2 I - T^{\sim}(s)T(s)$$
$$T(s) = \gamma I - \gamma^{-1}B'P(sI-A)^{-1}$$

where P is the stabilizing solution of the Riccati equation (with unknown P). Then, letting $\gamma\Omega(s)=T(s)$ it follows

$$I_\gamma(G) = -\frac{\gamma^2}{2\pi}\int_{-\infty}^{+\infty}\log\det\left[\Omega(-j\omega)'\Omega(j\omega)\right]d\omega = \lim_{z\to\infty} -\frac{\gamma^2}{2\pi}\int_{-\infty}^{+\infty}\log\det\left[\Omega(-j\omega)'\Omega(j\omega)\right]\frac{|z^2|}{|(z-j\omega)|^2}d\omega$$
$$= \lim_{z\to\infty} -\frac{\gamma^2}{\pi}\int_{-\infty}^{+\infty}\log\left|\det\Omega(j\omega)\right|\frac{|z^2|}{|(z-j\omega)|^2}d\omega$$
$$I_\gamma(G) = \lim_{z\to\infty} -\gamma^2 z\log\left|\det\Omega(z)\right| = \lim_{z\to\infty} -\gamma^2 z\log\left|\det\,(I-\gamma^{-2}B'P(zI-A)^{-1}B\right|$$
$$= \lim_{z\to\infty} -\gamma^2 z\log\left|\det\,(I-\frac{\gamma^{-2}}{z}B'P(B+M(z^{-1}))\right| = \lim_{z\to\infty} -\gamma^2 z\log\left|(1-\frac{\gamma^{-2}}{z}trace[B'P(B+M(z^{-1}))]\right|$$
$$= \lim_{z\to\infty} -\gamma^2 z\log\left|(1-\frac{\gamma^{-2}}{z}trace[B'PB]+O(z^{-2}))\right| = \lim_{z\to\infty} -\gamma^2 z(-\frac{\gamma^{-2}}{z}trace[B'PB]+O(z^{-2}))$$
$$= trace\left[B'PB\right]$$

In view of the Poisson integral formula it follows that

In the expressions above we set $M(z^{-1})=ABz^{-1}+A^2Bz^{-2}+\ldots$Moreover, $O(z^{-2})$ denotes terms of powers z^{-2}, z^{-3} ect.. Finally, the formulas $\det(I+\varepsilon V)=1+\varepsilon\,trace(V)+O(\varepsilon^2)$ and $\log\det(I+\varepsilon V)=\varepsilon\,trace(V)+O(\varepsilon^2)$ have been used. The proof of the proposition with the solution of the Riccati equation with unknown Q follows the same lines and therefore is omitted.

Complex stability radius for norm bounded uncertain systems

A_{un}=A+LΔN

A= stable (eigenvalues with strictly negative real part)

$$
\begin{aligned}
r_c(A,B,C) &= \inf\left\{\|\Delta\| : \Delta \in C^{m\times p} \ and \ A+B\Delta C \ is \ unstable\right\} \\
&= \inf_{s\in j\omega} \inf\left\{\|\Delta\| : \Delta \in C^{m\times p} \ and \ \det(sI - A - B\Delta C) = 0\right\} \\
&= \inf_{s\in j\omega} \inf\left\{\|\Delta\| : \Delta \in C^{m\times p} \ and \ \det(I - \Delta C(sI-A)^{-1}B) = 0\right\}
\end{aligned}
$$

Linear algebra problem: compute

$$
\mu_c(M) = \left[\inf\left\{\|\Delta\| : \Delta \in C^{m\times p} \ and \ \det(I - \Delta M) = 0\right\}\right]^{-1}
$$

Proposition

The complex stability radius of A+LDN is

$r_c(A,L,N)=(1/\|N(sI-A)^{-1}L\|_\infty$.

Definition
An uncertain system is quadratically stable if there exists P>0 (independent of the uncertain parameters) such that A_{un}'P+PA_{un}<0 for all uncertain parameters in the uncertainty set.

Proposition
Let A_{un}=A+LΔN and ||Δ||≤α. The system is quadratically stable if and only if $1/\|N(sI-A)^{-1}L\|_\infty < \alpha^{-1}$.

Real stability radius

A= stable (eigenvalues with strictly negative real part)

Linear algebra problem: compute

$$
\begin{aligned}
r_r(A,B,C) &= \inf\{\|\Delta\| : \Delta \in R^{m\times p} \; and \; A + B\Delta C \; is \; unstable\} \\
&= \inf_{s\in j\omega} \inf\{\|\Delta\| : \Delta \in R^{m\times p} \; and \; \det(sI - A - B\Delta C) = 0\} \\
&= \inf_{s\in j\omega} \inf\{\|\Delta\| : \Delta \in R^{m\times p} \; and \; \det(I - \Delta C(sI - A)^{-1}B) = 0\}
\end{aligned}
$$

$$
\mu_r(M) = \left[\inf\{\|\Delta\| : \Delta \in R^{m\times p} \; and \; \det(I - \Delta M) = 0\}\right]^{-1}
$$

Proposition

$$
\mu_r(M) = \inf_{\gamma\in(0,1]} \sigma_2\left(\begin{bmatrix} \operatorname{Re} M & -\gamma \operatorname{Im} M \\ \gamma^{-1} \operatorname{Im} M & \operatorname{Re} M \end{bmatrix}\right)
$$

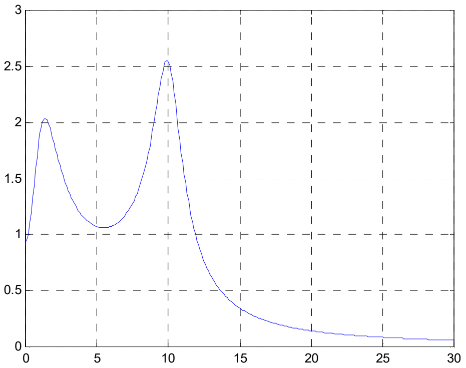
Taking M=G(s)=C(sI-A)$^{-1}$B it follows

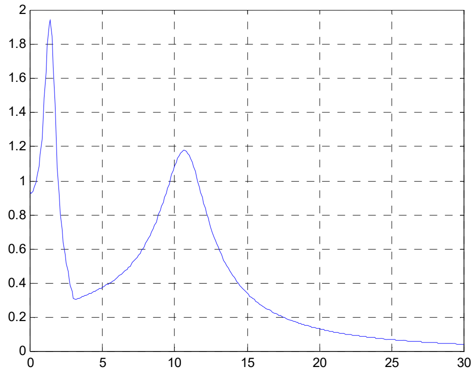
$$
r_r(A,B,C)^{-1} = \sup_{\omega} \inf_{\gamma\in(0,1]} \sigma_2\left(\begin{bmatrix} \operatorname{Re} G(j\omega) & -\gamma \operatorname{Im} G(j\omega) \\ \gamma^{-1} \operatorname{Im} G(j\omega) & \operatorname{Re} G(j\omega) \end{bmatrix}\right)
$$

Example

$$A=\begin{bmatrix}79 & 20 & -30 & -20\\ -41 & -12 & 17 & 13\\ 167 & 40 & -60 & -38\\ 33.5 & 9 & -14.5 & -11\end{bmatrix},\quad B=\begin{bmatrix}0.2190 & 0.9347\\ 0.0470 & 0.3835\\ 0.6789 & 0.5194\\ 0.6793 & 0.8310\end{bmatrix}$$

$$C=\begin{bmatrix}0.0346 & 0.5297 & 0.0077 & 0.0668\\ 0.0535 & 0.6711 & 0.3834 & 0.4175\end{bmatrix}$$





$$\Delta_{worst}=\begin{bmatrix}-0.4996 & 0.1214\\ 0.1214 & 0.4996\end{bmatrix}$$

Polytopic uncertainty

$$A_{un}=\sum_{i=1}^{M}A_i\sigma_i,\quad \sigma_i\ge 0,\ \sum_{i=1}^{M}\sigma_i=1$$

Proposition

The system is quadratically stable if and only if there exists P>0 such that

$$A_i'P+PA_i<0,\quad \forall i$$

Proof

$A_i'P+PA_i<0,\ \forall i$ implies $\left(\sum_{i=1}^{M}A_i'\sigma_i\right)P+P\left(\sum_{i=1}^{M}A_i\sigma_i\right)<0$ for all σ_i in the simplex. Viceversa, if there exist P>0 such that $\left(\sum_{i=1}^{M}A_i'\sigma_i\right)P+P\left(\sum_{i=1}^{M}A_i\sigma_i\right)<0$ for all σ_i in the simplex, then in particular $A_i'P+PA_i<0,\ \forall i$.

Time-varying

Proposition

The system is robustly stable if there exists $P_i>0$, G, V such that

$$\begin{bmatrix} A_i'G+G'A_i & P_i+A_i'V-G' \\ P_i+V'A_i-G & -V-V' \end{bmatrix}<0\quad \forall i$$

Proof

From the assumption it follows that

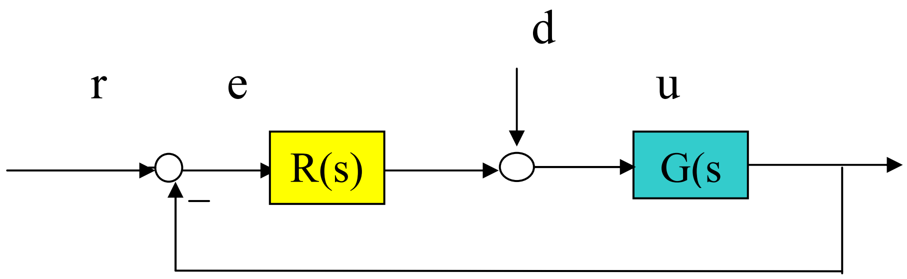
$$\begin{bmatrix} \left(\sum_{i=1}^{M}A_i'\sigma_i\right)G+G'\left(\sum_{i=1}^{M}A_i\sigma_i\right) & \left(\sum_{i=1}^{M}P_i\sigma_i\right)+\left(\sum_{i=1}^{M}A_i'\sigma_i\right)V-G' \\ \left(\sum_{i=1}^{M}P_i\sigma_i\right)+V'\left(\sum_{i=1}^{M}A_i\sigma_i\right)-G & -V-V' \end{bmatrix}<0$$

By multiplying by [I A_i'] on the left and by [I A_i'] ' on the right it follows

$$\left(\sum_{i=1}^{M}A_i'\sigma_i\right)\left(\sum_{i=1}^{M}P_i\sigma_i\right)+\left(\sum_{i=1}^{M}P_i\sigma_i\right)\left(\sum_{i=1}^{M}A_i\sigma_i\right)<0$$

constant

Stability of feedback systems



The problem consists in the analysis of the asymptotic stability (internal stability) of closed-loop system from some characteristic transfer functions. This result is useful also for the design.

Theorem

The closed loop system is asymptotically stable if and only if the transfer matrix B(s) from the input to the output

$$\begin{bmatrix} r \\ d \end{bmatrix} \xrightarrow{B(s)} \begin{bmatrix} e \\ u \end{bmatrix}$$

is stable.

$$B(s) = \begin{bmatrix} (I + G(s)R(s))^{-1} & -(I + G(s)R(s))^{-1}G(s) \\ (I + R(s)G(s))^{-1}R(s) & (I + R(s)G(s))^{-1} \end{bmatrix}$$

Stability of interconnected SISO systems

The closed loop system is asymptotically stable if and only if the two transfer functions

$$V(s)=\frac{G(s)}{1+R(s)G(s)},\ M(s)=\frac{G(s)}{1+R(s)G(s)}$$

are stable.

- *Notice that if V(s), M(s), are stable, then also the complementary sensitivity function and the sensitivity function are stable.*

- *The stability of the two transfer functions are made to avoid prohibited (unstable) cancellations between C(s) and P(s).*

Example

Let R(s)=(s-1)/(s+1), G(s)=1/(s-1) . Then S(s)=(s+1)/(s+2), V(s)=(s-1)/(s+2) e M(s)=(s+1)/(s^2+s-2)

Comments

The proof of the Theorem above can be carried out pursuing different paths. A simple way is as follows. Let assume that R(s) and G(s) are described by means of *minimal realizations* G(s)=(A,B,C,D), R(s)=(F,G,H,E). It is possible to write a realization of the closed-loop system as $\Sigma_{cl}=(A_{cl},B_{cl},C_{cl},D_{cl})$. Obviously, if A_{cl} is asymptotically stable, then B(s) is stable. Vice-versa, if B(s) is stable, asymptotic stability of A_{cl} follows form being $\Sigma_{cl}=(A_{cl},B_{cl},C_{cl},D_{cl})$ a minimal realization. This can be seen through the well known PBH test.

In our example, it follows

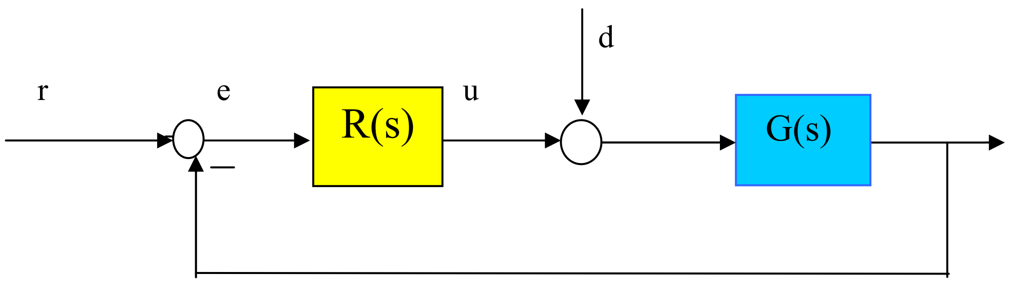
$$A_{cl}=\begin{bmatrix}0 & -2\\ -1 & -1\end{bmatrix},\quad B_{cl}=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix},\quad C_{cl}=\begin{bmatrix}-1 & 0\\ -1 & -2\end{bmatrix},\quad D_{cl}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

Hence, A_{cl} is unstable and the closed-loop system is reachable and observable. This means that G(s) is unstable as well.

$$B(s)=\begin{bmatrix}\dfrac{s+1}{s+2} & \dfrac{-(s+1)}{(s-1)(s+2)}\\ \dfrac{s-1}{s+2} & \dfrac{s+1}{s+2}\end{bmatrix}\overset{SMM}{\triangleright}\begin{bmatrix}\dfrac{1}{(s-1)(s+2)} & 0\\ 0 & (s+1)(s-1)\end{bmatrix}$$

Parametrization of stabilizing controllers

1° case: SISO systems and P(s) stable



Theorem

The family of all controllers C(s) such that the closed-loop system is asymptotically stable is:

$$R(s) = \frac{Q(s)}{1 - G(s)Q(s)}$$

where Q(s) is proper and stable and $Q(\infty)G(\infty) \neq 1$.

Proof of Theorem

Let R(s) be a stabilizing controller and define Q(s)=R(s)/(1+R(s)P(s)). Notice that Q(s) is stable since it is the transfer function from r to u. Hence C(s) can be written as =Q(s)/(1-G(s)Q(s)), with Q(s) stable and, obviously, the stability of both Q(s) and G(s) implies that $Q(\infty)G(\infty)=R(\infty)/(1+R(\infty)G(\infty)) \neq 1$.

Vice-versa, assume that Q(s) is stable and $Q(\infty)G(\infty)\neq 1$. Define R(s)=Q(s)/(1-G(s)Q(s)). It results that

S(s)=1/(1+R(s)G(s))=1-G(s)Q(s),

R(s)=R(s)/(1+R(s)G(s))=Q(s),

H(s)=G(s)/(1+R(s)G(s))=G(s)(1-G(s)Q(s))

are stable. This means that the closed-loop system is asymptotically stable.

Parametrization of stabilizing controllers

2° case: SISO systems and generic P(s)

It is always possible to write (coprime factorization)

$$P(s) = \frac{N(s)}{D(s)}$$

where N(s) and D(s) are stable rational coprime functions, i.e. such that there exist two stable rational functions X(s) e Y(s) satisfying (equation of Diofanto, Aryabatta, Bezout)

$$N(s)X(s) + D(s)Y(s) = 1$$

Theorem

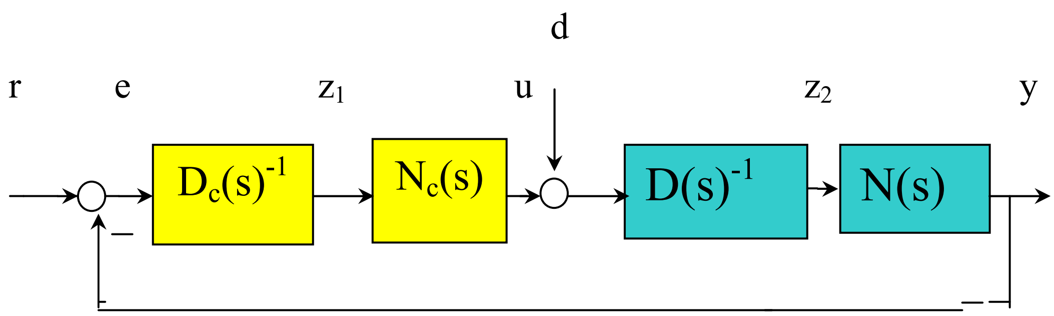
The family of all controllers C(s) such that the closed-loop system is well-posed and asymptotically stable is:

$$R(s) = \frac{X(s) + D(s)Q(s)}{Y(s) - N(s)Q(s)}$$

where Q(s) is proper, stable and such that $Q(\infty)N(\infty) \neq Y(\infty)$.

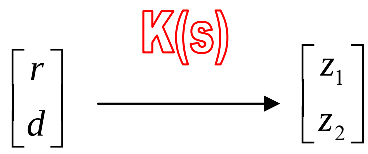
Comments

The proof of the previous theorem can be carried out following different ways. However, it requires a preliminary discussion on the concept of coprime factorization and on the stability of factorized interconnected systems.



Lemma

Let P(s)=N(s)/D(s) e C(s)=N_c(s)/D_c(s) stable coprime factorizations. Then the closed-loop system is asymptotically stable if and only if the transfer matrix K(s) from [r′ d′]′ to [z_1′ z_2′]′ is stable.



$$K(s)=\begin{bmatrix} D_c(s) & N(s) \\ -N_c(s) & D(s) \end{bmatrix}^{-1}$$

Proof of Lemma

Define four stable functions X(s),Y(s),X_c(s),Y_c(s) such that

$$X(s)N(s)+Y(s)D(s)=1$$

$$X_c(s)N_c(s)+Y_c(s)D_c(s)=1$$

To proof the Lemma it is enough to resort to Theorem 1, by noticing that the transfer functions K(s) and B(s) (from [r′ d′]′ to [e′ u′]′) are related as follows:

$$K(s)=\begin{bmatrix} Y_c(s) & X_c(s) \\ -X(s) & Y(s) \end{bmatrix}G(s)-\begin{bmatrix} 0 & X_c \\ -X & 0 \end{bmatrix} \qquad B(s)=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}+\begin{bmatrix} 0 & -N \\ N_c & 0 \end{bmatrix}K(s)$$

Proof of Theorem

Assume that Q(s) is stable and Q(∞)N(∞) ≠ Y(∞). Moreover, define R(s)=(X(s)+D(s)Q(s))/(Y(s)-N(s)Q(s)). It follows that

1=N(s)X(s)+D(s)Y(s)=N(s)(X(s)+D(s)Q(s))+D(s)(Y(s)-N(s)Q(s))

so that the functions X(s)+D(s)Q(s) e Y(s)-N(s)Q(s) defining R(s) are coprime (besides being both stable). Hence, the three characterist transfer functions are:

S(s)=1/(1+R(s)G(s))=D(s)(Y(s)-N(s)Q(s)),

V(s)=R(s)/(1+R(s)G(s))=D(s)(X(s)+D(s)Q(s)),

D(s)=G(s)/(1+R(s)G(s))=N(s) (Y(s)-N(s)Q(s))

Since they are all stable, the closed-loop system is asymptotically stable as well.

Vice-versa, assume that R(s)=N_c(s)/M_c(s) (stable coprime factorization) is such that the closed-loop system is well-posed and asymptotically stable. Define Q(s)=(Y(s)N_c(s)-X(s)D_c(s))/(D(s)M_c(s)+N(s)N_c(s)). Since the closed-loop system is asymptotically stable and (N_c,D_c) are coprime, then, in view of the Lemma it follows that

Q(s) = (Y(s)N_c(s)-X(s)D_c(s))/(D(s)D_c(s)+N(s)N_c(s)) = [0 I]K(s)[Y(s)′-X(s)′]′

is stable. This leads to C(s)=(X(s)+D(s)Q(s))/(Y(s)-N(s)Q(s)). Finally, Q(∞)N(∞) ≠ Y(∞), as can be easily be verified.

Coprime Factorization

1° case: SISO systems

Lemma 1 makes reference to a factorized description of a transfer function. Indeed, it is easy to see that it is always possible to write a transfer function as the ratio of two stable transfer functions without common divisors (coprime). For example

$$G(s) = \frac{s+1}{(s-1)(s+10)(s-2)} = N(s)D(s)^{-1}$$

with

$$N(s) = \frac{1}{(s+10)(s+1)}, \quad D(s) = \frac{(s-1)(s-2)}{(s+1)^2}$$

It results

$$N(s)X(s) + D(s)X(s) = 1$$

with

$$X(s) = \frac{4(16s-5)}{(s+1)}, \quad Y(s) = \frac{(s+15)}{(s+10)}$$

Euclide's Algorithm

Coprime Factorization
SISO systems

Lemma 1 makes reference to a factorized description of a transfer function. Indeed, it is easy to see that it is always possible to write a transfer function as the ratio of two stable transfer functions without common (not unimodular) divisors (coprimeness in H_∞). For example

$$G(s) = \frac{s+1}{(s-1)(s+10)(s-2)} = N(s)D(s)^{-1}$$

with

$$N(s) = \frac{1}{(s+10)(s+1)}, \quad D(s) = \frac{(s-1)(s-2)}{(s+1)^2}$$

It results

$$N(s)X(s) + D(s)Y(s) = 1$$

with

$$X(s) = \frac{4(16s-5)}{(s+1)}, \quad Y(s) = \frac{(s+15)}{(s+10)}$$

Euclide's Algorithm

Coprime Factorization
MIMO systems

In the MIMO case, we need to distinguish between right and left factorizations.

Right and left factorization

Given $G(s)$, find four proper and stable transfer matrices:

$$G(s) = N_r(s)D_r(s)^{-1} = D_l(s)^{-1}N_l(s)$$

$$N_r(s),\, D_r(s) \quad \textit{right} \quad \textit{coprime} \quad X_r(s)N_r(s) + Y_r(s)D_r(s) = I$$

$$N_l(s),\, D_l(s) \quad \textit{left} \quad \textit{coprime} \quad N_l(s)X_l(s) + D_l(s)Y_l(s) = I$$

$$G(s) = C(sI - A)^{-1}B + D$$

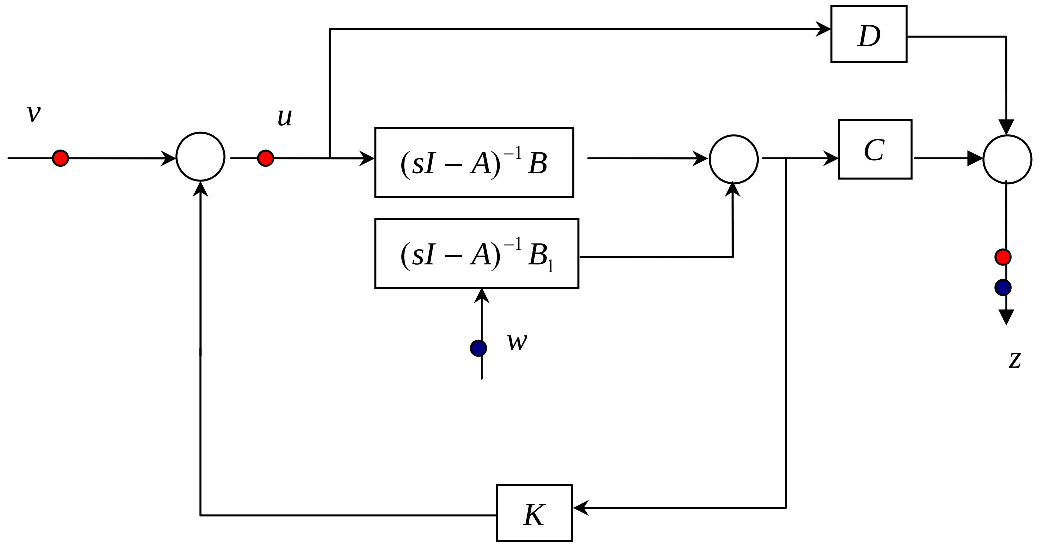
Choose K and L such that $A+BK$ and $A+LC$ are Hurvitz. Then:

$$N_r(s) = D + (C + DK)(sI - A - BK)^{-1}B$$
$$D_r(s) = I + K(sI - A - BK)^{-1}B$$
$$N_l(s) = D + C(sI - A - LC)^{-1}(B + LD)$$
$$D_l(s) = I + C(sI - A - LC)^{-1}L$$

Note: (i) the set of all matrices K such that $A+BK$ is stable is given by $K = WS^{-1}$ where $S > 0$ and W solve the LMI $AS + BW + W'B' + SA' < 0$. (ii) the set of all matrices L such that $A+LC$ is stable is given by $L = P^{-1}\Omega$ where $P > 0$ and Ω solve the LMI $A'P + C'\Omega' + \Omega C + PA < 0$.

Interpretation – state-feedback control

$$\dot{x} = Ax + Bu + B_1 w$$
$$z = Cx + Du$$
$$y = Kx + v$$

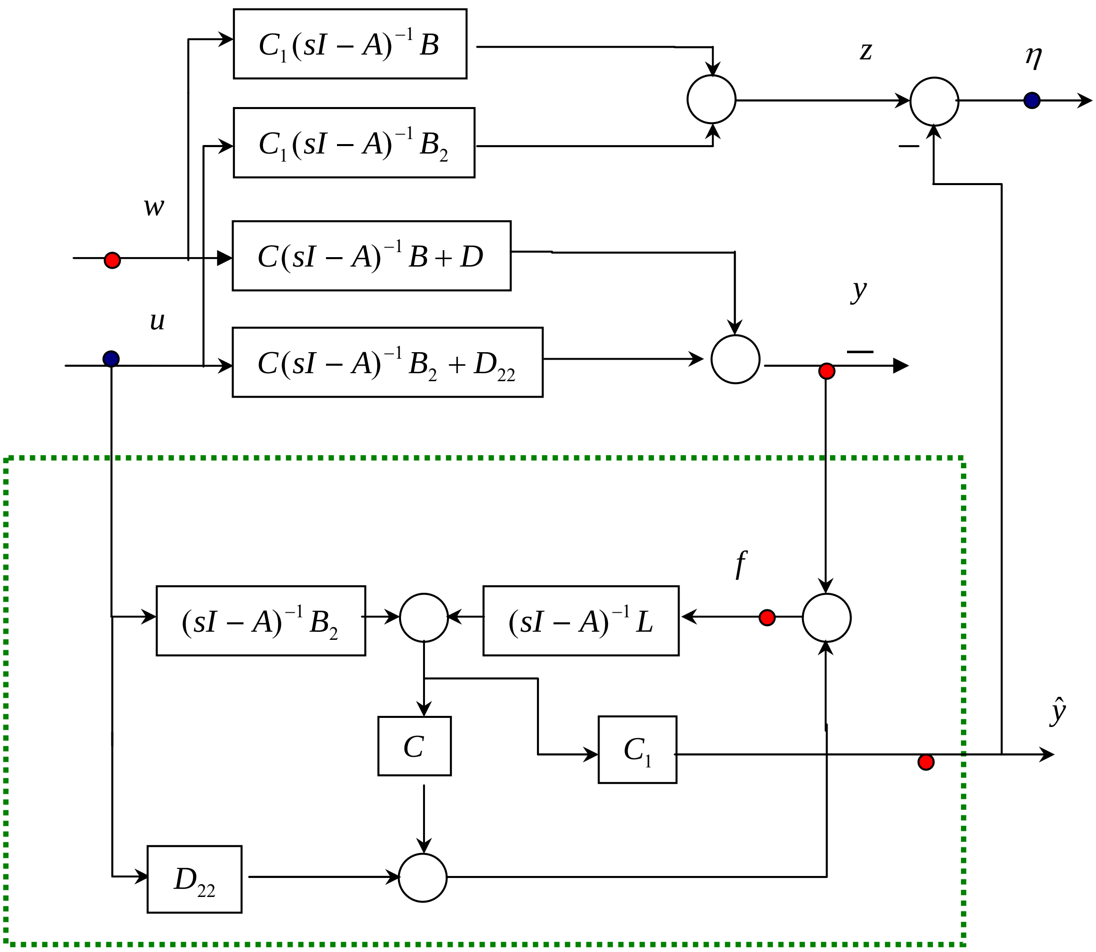


The transfer function from v to z is $N_r(s)$. The transfer function from v to u is $D_r(s)$. The design matrix K is such that $A+BK$ is Hurwitz. Further requirements can be posed on the norm of the transfer function from the disturbance w to the performance variable z.

Interpretation – output injection

$$\dot{x} = Ax + Bw + B_2 u$$
$$y = Cx + Dw + D_{22} u$$
$$z = C_1 x$$

$$\dot{\hat{x}} = A\hat{x} + B_2 u + L(C\hat{x} + D_{22}u - y)$$
$$\hat{y} = C\hat{x}$$



The transfer function from w to f is $N_l(s)$. The transfer function from $-y$ to $\hat{y}$ is $D_l(s)$.

Double coprime factorization

Given P(s), find eight proper and stable transfer functions such that:

$$(i) \quad P(s) = N_r(s)D_r(s)^{-1} = D_L(s)^{-1}N_L(s)$$

$$(ii) \quad \begin{bmatrix} Y_r(s) & X_r(s) \\ -N_l(s) & D_l(s) \end{bmatrix} \begin{bmatrix} D_r(s) & -X_l(s) \\ N_r(s) & Y_l(s) \end{bmatrix} = I$$

Hermite's algorithm

Observer and control law

Construction of a stable double coprime factorization

Let (A,B,C,D) a stabilizable and detectable realization of G(s) and conventionally write G=(A,B,C,D) .

Theorem

Let K e L two matrices such that A+BK e A+LC are stable. Then there exists a stable double coprime factorization, given by:

D_r=(A+BK,B, K,I),	N_r=(A+BK,B,C+DK,D)
Y_l=(A+BK,L,-C-DK,I),	X_l=(A+BK,L,K,0)
D_l=(A+ LC,L,C,I),	N_l=(A+LC,B+LD,C,D)
Y_r=(A+LC,B+LD, -K,I),	X_r=(A+LC,L,K,0)

Proof of Theorem

The existence of stabilizing K e L is ensured by the assumptions of stabilizability and detectability of the system. To check that the 8 transfer functions form a stable double coprime factorization notice first that they are all stable and that M_d e M_s are biproper systems. Finally one can use matrix calculus to verify the theorem. Alternatively, suitable state variables can be introduced. For example, from

$$\dot{x} = Ax + Bu = (A + BK)x + Bv$$
$$y = Cx + Du = (C + DK)x + Dv$$
$$u = Fx + v \quad control \quad law$$

one obtains y=G(s)u=N_r(s)v, u=D_r(s)v, so that G(s)D_d(s)=N_d(s).

Similarly,

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$\begin{cases} \dot{\theta} = A\theta + Bu + L\eta \\ \eta = C\theta + Du - y \end{cases} \quad observer$$

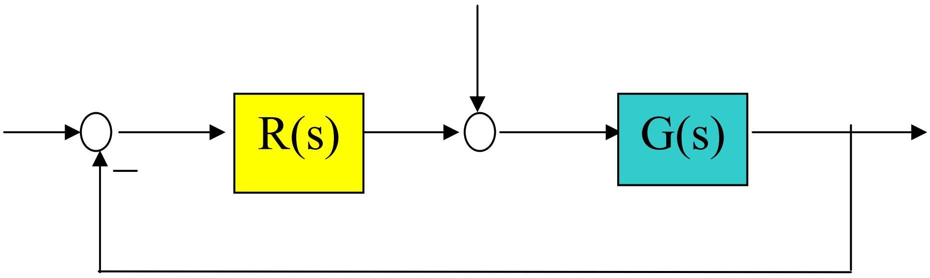
implies η=N_l(s)u-D_l(s)y=N_l(s)u-F_l(s)G(s)y=0 (stable autonomous dynamic) so that N_l(s)=D_l(s)G(s). Analoguously one can proceed for the rest of the proof.

Parametrization of stabilizing controllers

3° case: MIMO systems and generic P(s)

$(i) \quad P(s) = N_r(s)D_r(s)^{-1} = D_l(s)^{-1}N_l(s)$

$$(ii) \quad \begin{bmatrix} Y_r(s) & X_r(s) \\ -N_l(s) & D_l(s) \end{bmatrix} \begin{bmatrix} D_r(s) & -X_l(s) \\ N_r(s) & Y_l(s) \end{bmatrix} = I$$

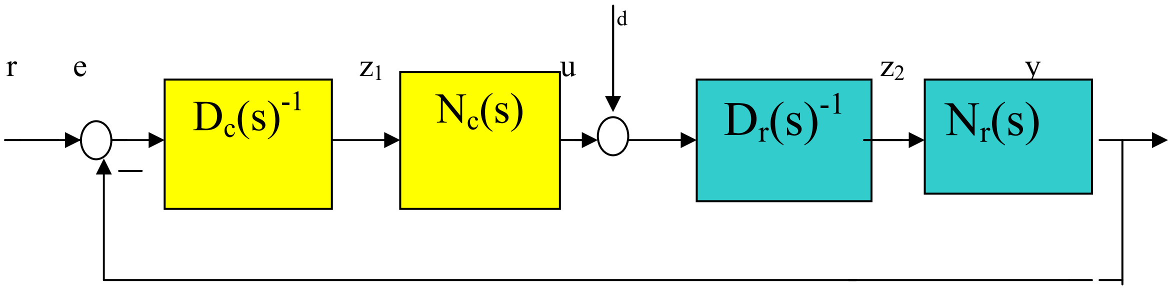


Theorem

The family of all proper transfer matrices R(s) such that the closed-loop system is well-posed and asymptotically stable is:

$$R(s) = (X_s(s) + D_r(s)Q_r(s))(Y_L(s) - N_r(s)Q_r(s))^{-1}$$
$$= (Y_r(s) - Q_l(s)N_l(s))^{-1}(X_r(s) + Q_l(s)D_l(s))$$

where $Q_r(s)$ [$Q_l(s)$] is stable and such that $N_r(\infty)Q_r(\infty) \neq Y_l(\infty)$ [$Q_l(\infty)N_l(\infty) \neq Y_r(\infty)$].



Lemma

Let $G(s)=N_r(s)D_r(s)^{-1}$ and $G(s)=N_c(s)D_c(s)^{-1}$ stable right coprime factorizations. Then the closed-loop system is asymptotically stable if and only if the transfer matrix K(s) from $[r' \ d']'$ to $[z_1' \ z_2']'$ is stable.

$$K(s) = \begin{bmatrix} D_c(s) & N_r(s) \\ -N_c(s) & D_r(s) \end{bmatrix}^{-1}$$

Proof of Lemma

It is completely similar to the proof of a previous Lemma (case 2).

Proof of Theorem

Assume that Q(s) is stable and define

$R(s)=(X_s(s)+D_r(s)Q(s))(Y_L(s)-N_r(s)Q(s))^{-1}$

It results that

$I=N_l(s)X_l(s)+D_l(s)Y_l(s)$

$=N_l(s)(X_l(s)+D_r(s)Q(s))+D_l(s)(Y_l(s) - N_r(s)Q(s))$

so that the functions $X_l(s)+D_r(s)Q(s)$ e $l_L(s)-N_r(s)Q(s)$ defining R(s) are right coprime (besides being both stable). The four transfer matrices characterizing the closed-loop are:

$(1+GR)^{-1}= (Y_l-N_rQ)D_s$

$-(1+GR)^{-1}G= (Y_l-N_rQ)N_l$

$(1+RG)^{-1}R=R(I+PC)^{-1}= (X_l+D_rQ)D_l$

$(1+RG)^{-1}=I-R(I+PR)^{-1}R= I-(X_l+D_rQ)N_l$

They are all stable so that the closed-loop system is asymptotically stable.

Vice-versa, assume that $R(s)=N_c(s)D_c(s)^{-1}$ (right coprime factorization) is stabilizing. Notice that the matrices

$$\begin{bmatrix} Y_r(s) & X_r(s) \\ -N_l(s) & D_l(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} D_r(s) & -N_c(s) \\ N_r(s) & D_c(s) \end{bmatrix}$$

have stable inverse. Hence,

$$\begin{bmatrix} Y_r(s) & X_d(s) \\ -N_l(s) & D_l(s) \end{bmatrix}\begin{bmatrix} D_r(s) & -N_c(s) \\ N_r(s) & D_c(s) \end{bmatrix} = \begin{bmatrix} I & -Y_r(s)N_c(s)X_r(s)D_c(s) \\ 0 & N_l(s)N_c(s)+D_l(s)M_c(s) \end{bmatrix} \quad (*)$$

has stable inverse as well. Then,

$Q(s)=(Y_r(s)N_c(s)-X_r(s)D_c(s))(D_l(s)D_c(s)+N_l(s)N_c(s))^{-1}$

is well–posed and stable. Premultiplying equation (*) by

$[M_r(s) - X_l(s); N_r(s)\ Y_l(s)]$

it follows

$N_c(s)M_c(s)^{-1}=(X_l(s)+M_r(s)Q_r(s))(Y_l(s)-N_r(s)Q_r(s))^{-1}$.

Observation

Taking Q(s)=0 we have the so-called *central controller*

$$R_0(s) = X_l(s)Y_l(s)^{-1}$$

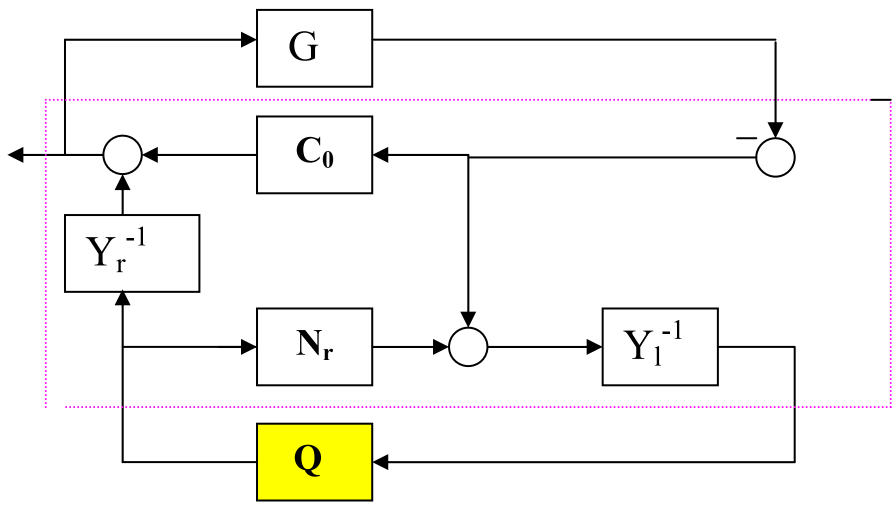
which coincides with the controller designed with the pole assignment technique.

Taking r=d=0 one has:

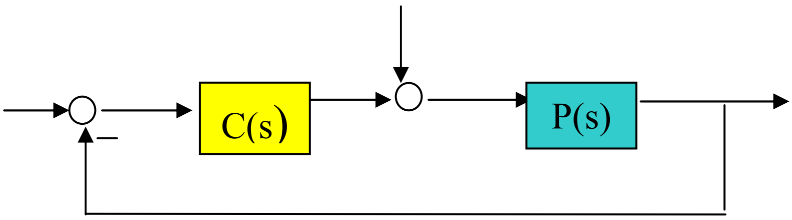
$$\dot{\xi} = A\xi + Bu + L(C\xi + Du - y)$$
$$u = F\xi$$

Lemma

$$R(s) = R_0(s) + Y_r(s)^{-1}Q(s)\left[(I - Y_l(s)^{-1}N_r(s)Q(s))\right]^{-1}Y_l(s)^{-1}$$



Strong Stabilization



The problem is that of finding, if possible, a stable controller which stabilizes the closed-loop system.

Example

$$P(s) = \frac{s-1}{(s-2)(s+2)}$$

is not stabilizable with a stable controller. Why?

$$P(s) = \frac{s-2}{(s-1)(s+2)}$$

is stabilizable with a stable controller. Why?

Stabilizazion of many plants

Two-step stabilization

Interpolation

Let consider a SISO control system with P(s)=N(s)/M(s) e C(s)=N_c(s)/M_c(s) (stable coprime stabilization). Then the closed-loop system is asymptotically stable if and only if U(s)=N(s)N_c(s)+M(s)M_c(s) is an unity (stable with stable inverse).

Since we want that C(s) be stable, we can choice N_c(s)=C(s) and D_c(s)=1. Then,

$$C(s) = \frac{U(s) - M(s)}{N(s)}$$

Of course, we must require that C(s) be stable. This fact depends on the role of the right zeros of P(s). Indeed, if N(b)=0, with Re(b)≥0 (b can be infinity), then the interpolation condition must hold true:

$$U(b) = M(b)$$

Consider the first example and take M(s)=(s-2)(s+2)/$(s+1)^2$, N(s)=(s-1)/$(s+1)^2$. Then, it must be: U(1)=-0.75, U(∞)=1. Obviously this is impossible.

Consider the second example and take M(s)=(s-1)/(s+2) e N(s)=(s-2)/$(s+2)^2$. It must be U(1)=0.25, U(∞)=1, which is indeed possible.

Parity interlacing property (PIP)

Theorem

- P(s) è strongly stabilizable if and only if the number of poles of P(s) between any pair of real right zeros of P(s) (including infinity) is an even number.

- We have seen that the PIP is equivalent to the existence of the existence of a unity which interpolates the right zeros of P(s). Hence, if the PIP holds, the problem boils down to the computation of this interpolant.

- In the MIMO case, Theorem 6 holds unchanged. However it is worth pointing out that the poles must be counted accordingly with their Mc Millan degree and the zeros to be considered are the so-called blocking zeros.

Optimal Linear Quadratic Control (LQ)
versus Full-Information H_2 control

$$\dot{x} = Ax + B\bar{u}, \quad x(0) = x_0, \quad J = \int_0^\infty (x'Qx + 2\bar{u}'Sx + \bar{u}'R\bar{u})dt$$

Assume that

$$R > 0, \quad H = \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$$

and notice that these conditions are equivalent to

$$R > 0, \quad W = Q - SR^{-1}S' \geq 0$$

Find (if any) u(·) minimizing J

Let C_{11} a factorization of $W=C_{11}'C_{11}$ and define

$$C_1 = \begin{bmatrix} C_{11} \\ R^{-1/2}S' \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$u = R^{1/2}\bar{u}, \quad z = C_1x + D_{12}u$$

Then, it is easy to verify that

$$J = \int_0^\infty (x'Qx + 2\bar{u}'Sx + \bar{u}'R\bar{u})dt = \int_0^\infty z'zdt = \|z\|_2^2$$

Moreover, the free motion of the state can be considered as a state motion caused by an impulsive input. Hence, with w(t)=imp(t), let

$$\begin{aligned} \dot{x} &= Ax + B_2u + B_1w \\ z &= C_1x + D_{12}u \\ x(0) &= 0 \end{aligned}$$

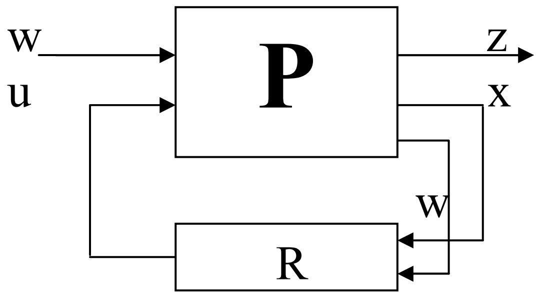
The (LQ) problem with stability is that of finding a controller

$$\begin{aligned} \dot{\xi} &= F\xi + G_1x + G_{21}w \\ u &= H\xi + E_1x + E_2w \end{aligned}$$

fed by x and w and yielding u that minimizes the H_2 norm of the transfer function from w to z.

LQS problem

Full information



$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{12} u$$
$$y = \begin{bmatrix} x \\ w \end{bmatrix}$$

Problem: Find the minimun value of $||T_{zw}||_2$ attainable by an admissible controller. Find an admissible controller minimizing $||T_{zw}||_2$. Find a set of all controllers generating all $||T_{zw}||_2<\gamma$.

Assumptions
(A,B_2) stabilizable
$D_{12}'D_{12}>0$
(A_c,C_{1c}) detectable

$A_c = A-B_2(D_{12}'D_{12})^{-1} D_{12}'C_1$
$C_{1c} = (I-D_{12} (D_{12}'D_{12})^{-1} D_{12}')C_1$

stable invariant zeros of (A,B2,C1,D12)

Solution of the Full Information Problem

Theorem

There exists an admissible controller FI minimizing $\|T_{zw}\|_2$. It is given by

$$F_2 = -(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1)$$

where P is the positive semidefinite and stabilizing solution of the Riccati equation

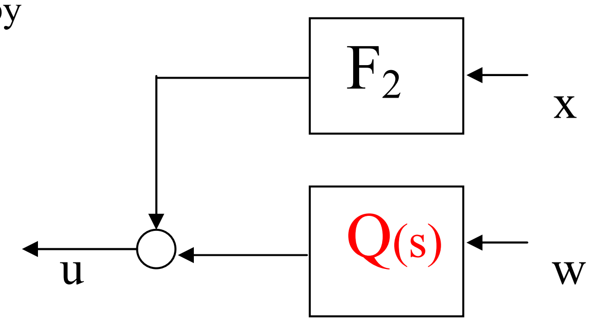
$$A'P + PA - (PB_2 + C_1'D_{12})(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1) + C_1'C_1 = 0$$

$$A_{cc} = A - B_2(D_{12}'D_{12}')^{-1}(B_2'P + D_{12}'C_1) = A + B_2F_2 \quad \textit{stable}$$

The minimum norm is:

$$\|T_{zw}\|_2^2 = \alpha^2, \quad \alpha = \|P_cB_1\|_2 = \sqrt{trace(B_1'PB_1)}, \quad P_c(s) = (C_1 + D_{12}F_2)(sI - A - B_2F_2)^{-1}$$

The set is given by

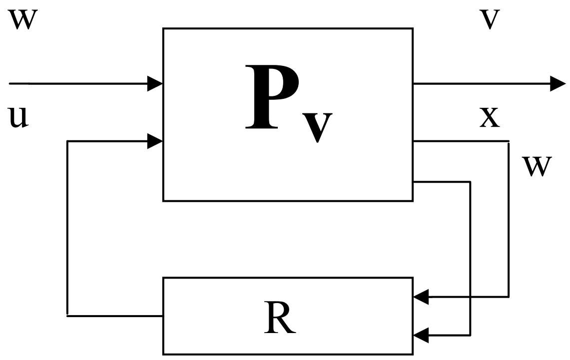


where Q(s) is a stable strictly proper system, satisfying

$$\|Q(s)\|_2^2 < \gamma^2 - \alpha^2$$

Proof of Theorem

The assumptions guarantee the existence of the stabilizing solution to the Riccati equation P. Let $v=u-F_2x$ so that $u=F_2x+v$, where v is a new input. Hence, $z(s)=P_c(s)B_1w(s)+U(s)v$, where $U(s)=P_c(s)B_2+D_{12}$. It follows that $T_{zw}(s)=P_c(s)B_1+U(s)T_{vw}(s)$. The problem is recast as to find a controller minimizing the norm from w to z of the following system



where P_v is given by:

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$v = -F_2 x + u$$
$$y = \begin{bmatrix} x \\ w \end{bmatrix}$$

Notice that $T_{zw}(s)$ is strictly proper iff $T_{vw}(s)$ is such. Exploiting the Riccati equality it is simple to verify that $U(s)^{\sim}U(s)=D_{12}'D_{12}$ and that $U(s)^{\sim}P_c(s)$ is antistable. Hence, $\|T_{zw}(s)\|_2^2=\|P_c(s)B_1\|_2^2 + \|T_{vw}(s)\|_2^2$. Hence the optimal control is $v=0$, i.e. $u=F_2x$.
Finally, take a controller K(s) such that $\|T_{zw}(s)\|_2^2 < \gamma^2$. From this controller and the system it is possible to form the transfer function $Q(s)=T_{vw}(s)$. Of course, it is $\|Q(s)\|_2^2 < \gamma^2-\alpha^2$. It is enough to show that the controller yielding $u(s)=F_2x(s)+v(s)=F_2x(s)+Q(s)w(s)$ generates the same transfer function $T_{zw}(s)$. This computation is left to the reader.

ACTUATOR DISTURBANCE

$$\dot{x} = Ax + B(w+u)$$
$$z = Cx + Du$$

The optimal H_2 state-feedback controller is

$$u = F_2 x, \quad F_2 = -(D'D)^{-1}(B'P + D'C)$$
$$0 = A'P + PA + C'C - F_2 D'DF_2$$

Proposition

The optimal H_2 control law ensures an H_∞ norm of the closed loop system (from w to z) lower than 2||D||.

Proof. Consider the equation of the Bounded Real Lemma for the closed-loop system, i.e.

$$(A + BF_2)'Q + Q(A + BF_2) + \gamma^{-2} QBB'Q + (C + DF_2)'(C + DF_2) < 0$$

Now take Q=P/α and consider both equations in P. It follows:

$$(1-\alpha^{-1})C'\left(I - D(D'D)^{-1}D'\right)C + PBRB'P < 0, \quad R = (D'D)^{-1}(1-\alpha^{-1}) + I\alpha^{-1}\gamma^{-2}$$

This condition is verified by choosing α≤1 and the minimum γ such that R≤0, i.e. the minimum γ such that

$$(\alpha - \alpha^2)I - \gamma^{-2}D'D \geq \alpha - \alpha^2 - \gamma^{-2}\bar{\sigma}^2(D) = 0$$

α-α^2

The term α-α^2 is maximized by α=0.5, for which α-α^2=0.25, so that the minimum γ is

$$\gamma = 2\bar{\sigma}(D)$$

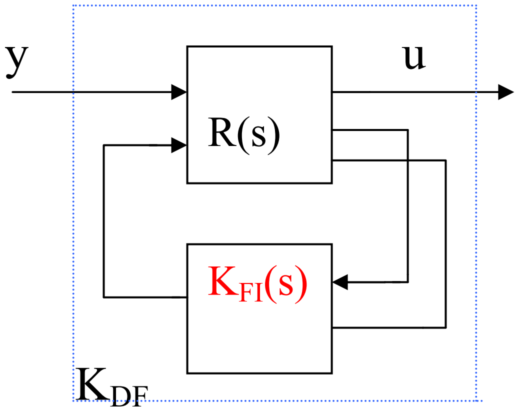
Disturbance Feedforward

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{12} u$$
$$y = C_2 x + w$$

Same assumptions as FI+stability of A-B_1C_2

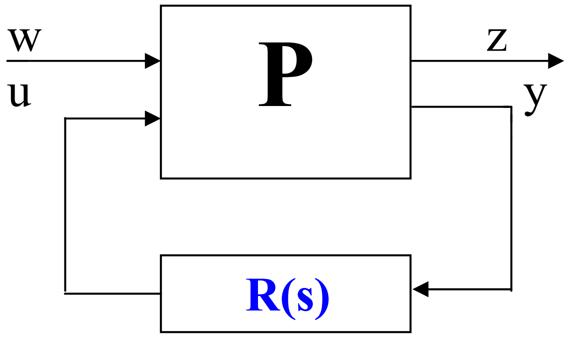
$C_2=0 \rightarrow$ direct compensation
$C_2\neq0 \rightarrow$ indirect compensation

$$R(s) = \left[\begin{array}{c|cc} A-B_1C_2 & B_1 & B_2 \\ \hline 0 & 0 & I \\ I & 0 & 0 \\ -C_2 & I & 0 \end{array}\right]$$

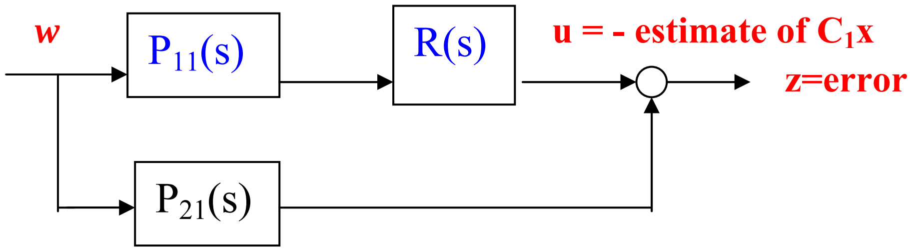


The proof of the DF theorem consist in verifying that the transfer function from w to z achieved by the FI regulator for the system A,B_1,B_2,C_1, D_{12} is the same as the transfer function from w to z obtained by using the regulator K_{DF} shown in the figure.

Output Estimation



$$\dot{x} = Ax + B_1 w + (B_2 u)$$
$$z = C_1 x + u$$
$$y = C_2 x + D_{21} w$$



Assumptions

(A,C_2) detectable

$D_{21}D_{21}' > 0$

$A_f = A - B_1 D_{21}'(D_{21}D_{21}')^{-1} C_2$

$B_{1f} = B_1(I - D_{21}' (D_{21}D_{21}')^{-1} D_{21})$

(A_f, B_{1f}) stabilizable

$(A - B_2 C_1)$ stable

Solution of the
Output Estimation problem

Theorem

There exists an admissible controller (filter) minimizing $\|T_{zw}\|_2$. It is given by

where

$$\dot{\xi} = A\xi + B_2u + L_2(C_2\xi - y)$$
$$u = -C_1\xi$$

$$L_2 = -(PC_2 + B_2D'_{21})(D_{21}D_{21})^{-1}$$

and P is the positive semidefinite and stabilizing solution of the Riccati equation

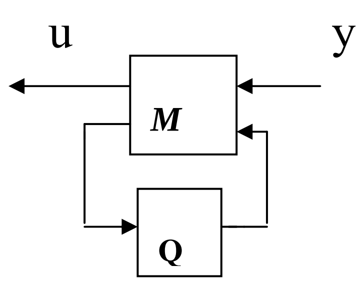
$$AP+PA'-(PC_2'+B_1D_{21}')(D_{21}D_{21}')^{-1}(C_2P+D_{21}B_1')+B_1B_1'=0$$
$$A_{ff}=A-(PC_2'+B_1D_{21}')(D_{21}D_{21}')^{-1}=A+L_2C_2 \quad stable$$

The optimal norm is

$$\|T_{zw}\|_2^2=\|C_1P_f(s)\|_2^2=\sqrt{trace(C_1PC_1')},\quad P_f(s)=(sI-A-L_2C_2)^{-1}(B_1+L_2D_{21})$$

The set of all controllers generating all $\|T_{zw}\|_2 < \gamma$ is given by

$$M(s)=\left[\begin{array}{c|cc} A_{ff}-B_2C_1 & L_2 & -B_2 \\ \hline C_1 & 0 & I \\ C_2 & I & 0 \end{array}\right]$$



Where Q(s) is proper, stable and such that $\|Q(s)\|_2^2 < \gamma^2 - \|C_1P_f(s)\|_2^2$

Proof of Theorem

It is enough to observe that the "transpose system" has the same structure as the system defining the DF problem. Hence the solution is the "transpose" solution of the DF problem. It is worth noticing that the H_2 solution does not depend on the particular linear combination of the state that one wants to estimate.

$$\dot{\lambda} = A'\lambda + C'_1\varsigma + C_2'\nu$$
$$\mu = B_1'\lambda + D_{21}'\nu$$
$$\vartheta = B_2'\lambda + \varsigma$$

Kalman filtering

Given the system

$$\dot{x} = Ax + \varsigma_1 + B_{22}u$$
$$\bar{y} = Cx + \varsigma_2$$

where [z′₁ z′₂]′ are zero mean white Gaussian noises with intensity

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}' & W_{22} \end{bmatrix}, \quad W_{22} > 0$$

find an estimate of the linear combination Sx of the state such as to minimize

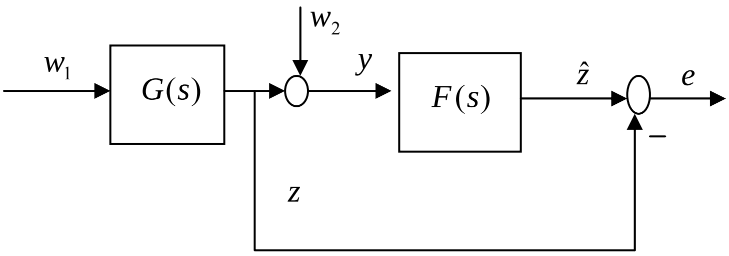
$$J = \lim_{t\to\infty} E\left[(Sx(t) - u(t))'(Sx(t) - u(t))\right]$$

Letting

$$C_1 = -S,\ C_2 = W_{22}{}^{-1}C,\ D_{21} = [0\ I],$$
$$B_{11}B_{11}' = W_{11} - W_{12}W_{22}{}^{-1}W_{12}', \quad B_1 = [B_{11}\ W_{12}W_{22}{}^{-1/2}]$$
$$y = W_{22}{}^{-1/2}\bar{y},\ z = C_1x + u,\ \begin{bmatrix} \varsigma_1 \\ \varsigma_2 \end{bmatrix} = \begin{bmatrix} B_{11} & W_{12}W_{22}{}^{-1/2} \\ 0 & W_{22}{}^{1/2} \end{bmatrix} w$$

the problem is recast to find a controller that minimizes the H_2 norm from w to z.

Remark on Wiener filtering (frequency domain)



Given G(s) strictly proper and stable, find F(s) stable such that the H_2 norm of the transfer function from $w=[w_1{}' \; w_2{}']'$ to e is minimized.

The transfer function from w to e is: $T_{ew} = \begin{bmatrix} FG-G & F \end{bmatrix}$ and hence

$$T_{ew}T_{ew}{}^{\sim} = F(GG^{\sim}+I)F^{\sim} - FGG^{\sim} - GG^{\sim}F^{\sim} + GG^{\sim} =$$

$$= (FG_o - GG^{\sim}G_o{}^{-1\sim})(G_o{}^{\sim}F^{\sim} - G_o{}^{-1}GG^{\sim}) + G(I - G^{\sim}G_o{}^{-1\sim}G_o{}^{-1}G)G^{\sim}$$

where G_o is stable with stable inverse such that $G_o G_o^{\sim} = I + GG^{\sim}$. Moreover

$$GG^{\sim}G_o{}^{-1\sim} = G_0 - G_o{}^{-1\sim} = G_0 - I + I - G_o{}^{-1\sim}$$

Applying the Pythagorean theorem, the optimal filter is given by

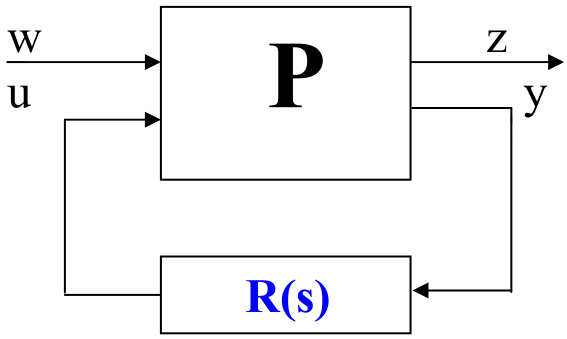
$$F^{ott}(s) = I - G_o(s)^{-1}$$

Comments

1) Letting $G(s)=C(sI-A)^{-1}B$, with A Hurwitz, and P the stabilizing solution of $AP+PA'-PBB'P+C'C=0$ it follows that $G_0(s)=I+C(sI-A)^{-1}PC'$ so that $G_o(s)^{-1}=I-C(sI-A+PCC')^{-1}PC'$ and $F^{ott}(s)= C(sI-A+PC'C)^{-1}PC'$, which corresponds to the optimal Kalman filter and optimal output estimator.
2) It results that
   $T_{ew}{}^{ott}T_{ew}{}^{ott\sim} = F^{ott}(GG^{\sim}+I)F^{ott\sim} - F^{ott}GG^{\sim} - GG^{\sim}F^{ott\sim} + GG^{\sim} = 2I - G_0{}^{-1\sim} - G_0{}^{-1}$ and since $\|G_0^{-1}\|_\infty \le 1$ it follows that $\|T_{ew}\|_\infty \le 2$.

The partial information problem

(LQG)



$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{12} u$$
$$y = C_2 x + D_{21} w$$

Assumptions: FI + DF.

Theorem

There exists an admissible controller minimizing $\|T_{zw}\|_2$. It is given by

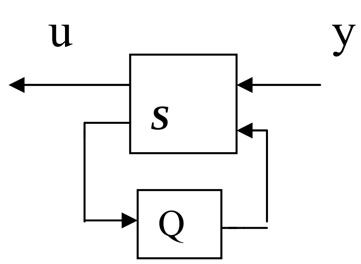
$$\dot{\xi} = A\xi + B_2 u + L_2(C_2\xi - y)$$
$$u = -F_2\xi$$

The optimal norm is

$$\|T_{zw}\|_2^2 = \|P_c(s)L_2\|_2^2 + \|C_1P_f(s)\|_2^2 = \|P_c(s)B_1\|_2^2 + \|F_2P_f(s)\|_2^2 = \gamma_0^2$$

The set of all controllers generating all $\|T_{zw}\|_2 < \gamma$ is given by

S(s)= $\left[\begin{array}{cc|cc} A+B_2F_2+L_2C_2 & & L_2 & -B_2 \\ \hline -F_2 & 0 & I & \\ C_2 & I & 0 & \end{array}\right]$



where Q(s) is proper, stable and such that $\|Q(s)\|_2^2 < \gamma^2 - \gamma_o^2$

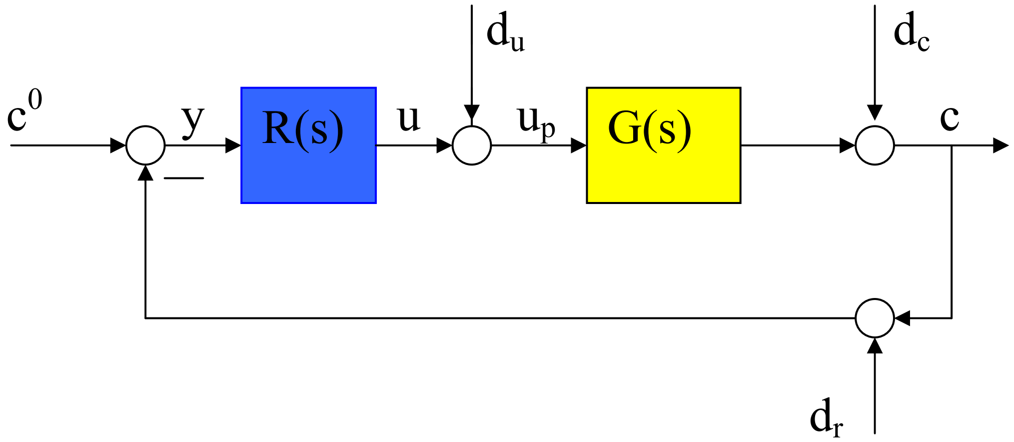
Proof

Separation principle: The problem is reformulated as an Output Estimation problem for the estimate of -F2 x.

Important topics to discuss

- Robustness of the LQ regulator (FI)
- Robustness of the Kalman filter (OE)
- Loss of robustness of the LQG regulator (Partial Information)
- LTR technique

The H_∞
design problem



Find G(s) in such a way to guarantee:

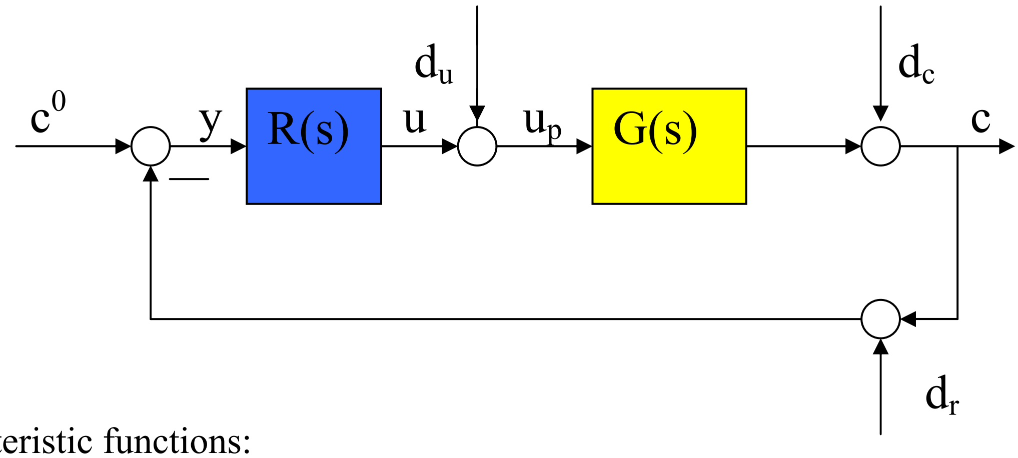
- Stability
- Satisfactory performances

Design in nominal conditions
Uncertainties description
Design under uncertaint conditions

Nominal Design

$G(s) = G_n(s)$

The performances are expressed in terms of requirements on some transfer functions



Characteristic functions:

- Sensitivity $S_n(s)=(I+G_n(s)R(s))^{-1}$
  $d_c \to c$, $c^0 \to y$, $-d_r \to y$

- Complementary sensitivity
  $F_n(s)=G_n(s)R(s)(I+G_n(s)R(s))^{-1}$
  $c^0 \to c$, $-d_r \to c$

- Control sensitivity
  $V_n(s)=R(s)(I+G_n(s)R(s))^{-1}$
  $c^0 \to u_p$, $-d_r \to u_p$, $-d_c \to u_p$

Shaping Functions

In the SISO case, the requirement to have a "small" transfer function $\phi(s)$ can be well expressed by saying that the absolute value $|\phi(j\omega)|$, for each frequency ω, must be smaller than a given function $\theta(\omega)$ (generally depending on the frequency):

Analogously, in the MIMO case, one can write

$$|\phi(j\omega)| < \theta(\omega), \quad \forall \omega$$
$$\bar{\sigma}[\phi(j\omega)] < \theta(\omega), \quad \forall \omega$$

This can be done by choosing a matrix transfer function W(s), stable with stable inverse ("shaping" function) , such that

$$\forall \omega \quad \bar{\sigma}[W^{-1}(j\omega)] = \theta(\omega) \leftarrow \|W(s)\phi(s)\|_\infty < 1$$

A general requirement is that the sensitivity function is small at low frequency (tracking) whereas the complementary sensitivity function is small at high frequency (feedback disturbances attenuation). Mixed sensitivity:

$$\left\| \begin{bmatrix} W_1(s)S_n(s) \\ W_2(s)T_n(s) \end{bmatrix} \right\|_\infty < 1$$

Description of the uncertainties

The nominal transfer function $G_n(s)$ of the process G(s) belongs to a set G which can be parametrized through a transfer function Δ(s) included in the set

For instance,

$$D_\alpha = \{\Delta(s) \mid \Delta(s) \in H_\infty, \quad \|\Delta(s)\|_\infty < \alpha \}$$

- G = {G(s) | G(s) = $G_n(s)+\Delta(s)$}
- G = {G(s) | G(s) = $G_n(s)(I+\Delta(s))$}
- G = {G(s) | G(s) = $(I+\Delta(s))G_n(s)$}
- G = {G(s) | G(s) = $(I-\Delta(s))^{-1}G_n(s)$}
- G = {G(s) | G(s) = $(I-G_n(s)\Delta(s))^{-1}G_n(s)$}

Examples

- $\mathcal{G} = \{G(s) \mid G(s) = G_n(s)+\Delta(s)\}$

Uncertaint unstable zeros

$$G(s) = \frac{s-2}{(s+2)(s+1)} + \frac{\varepsilon}{(s+2)(s+1)}$$

- $\mathcal{G} = \{G(s) \mid G(s) = G_n(s)(I+\Delta(s))\}$

Unmodelled high frequency poles or unstable zeros

$$G(s) = \frac{1}{(s+1)}\left(1 - \frac{\varepsilon s}{(1+\varepsilon s)}\right), \quad G(s) = \frac{1}{(s+1)}\left(1 - \frac{2}{(1+s)}\right)$$

- $\mathcal{G} = \{G(s) \mid G(s) = (I-\Delta(s))^{-1}G_n(s)\}$

Unmodelled unstable poles

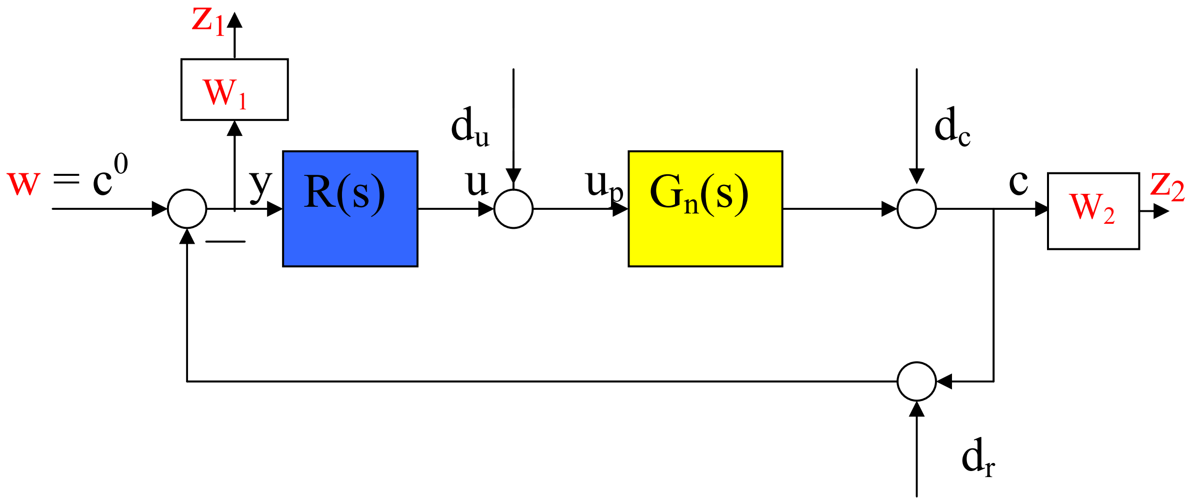
$$G(s) = \left(1 - \frac{10}{(1+s)}\right)^{-1} \frac{1}{(s+10)}$$

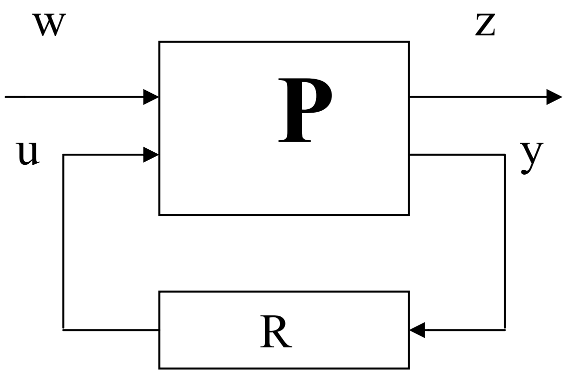
- $\mathcal{G} = \{G(s) \mid G(s) = (I-G_n(s)\Delta(s))^{-1}G_n(s)\}$

Uncertain unstable poles

$$G(s) = \left(1 - \frac{1}{s-1}\varepsilon\right)^{-1} \frac{1}{(s-1)}$$

Design in nominal conditions
Example of mixed performances





$$P = \begin{bmatrix} W_1 & -W_1 G_n \\ 0 & W_2 G_n \\ I & -G_n \end{bmatrix}$$

$W_5 = 1$

Computations in state-space.

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \; w = c^0$$

Let $d_u = d_c = d_r = 0$ and $G_n(s)$ be described by

$$\dot{x} = Ax + Bu$$
$$c = Cx$$

Let $W_1(s)$ be described by

$$\dot{\xi} = A_1\xi + B_1 y$$
$$z_1 = C_1\xi$$

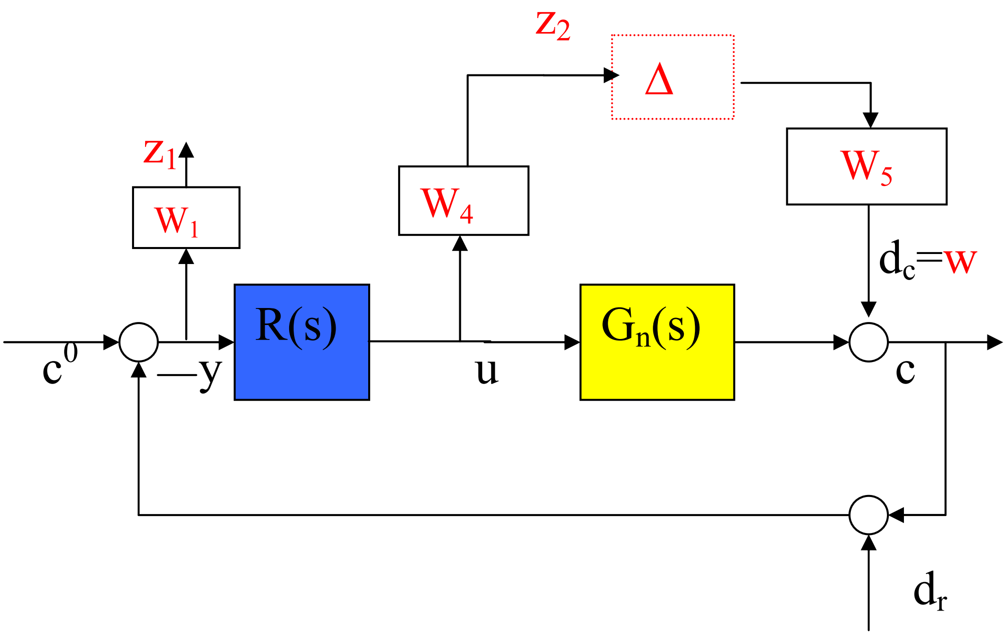
and $W_2(s)$ be described by

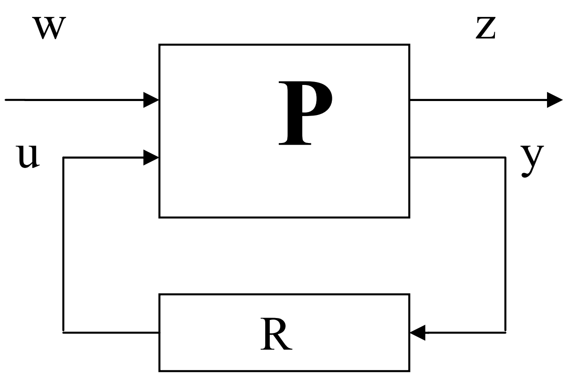
$$\dot{\eta} = A_2\eta + B_2 c$$
$$z_2 = C_2\eta$$

Then system P(s) is described by

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ -B_1C & A_1 & 0 \\ B_2C & 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ \xi \\ \eta \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \\ 0 \end{bmatrix} w + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} u$$
$$\begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix} = \begin{bmatrix} 0 & C_1 & 0 \\ 0 & 0 & C_2 \\ -C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \\ \eta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u$$

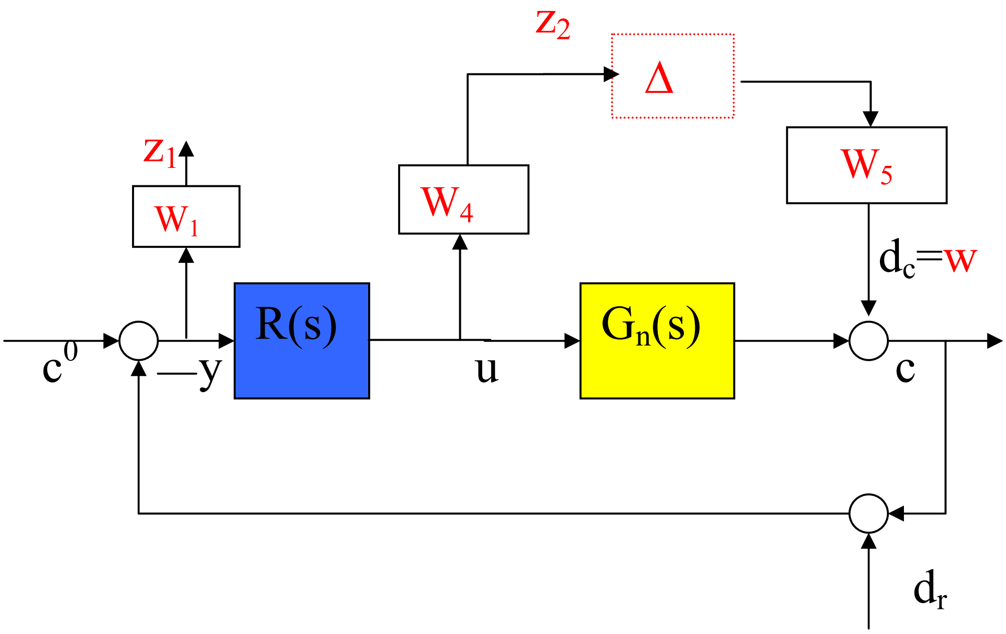
Design under uncertain condition: Robust stability and nominal performances





$$z=\begin{bmatrix} z_1 \\ z_2 \end{bmatrix},\ w=d_c \qquad P=\begin{bmatrix} -W_1 & -W_1G_n \\ 0 & W_4G_n \\ -I & -G_n \end{bmatrix}$$

Design under uncertain condition
Robust stability and robust performances



1) Robust sensitivity performance

$$\|W_1(s)S(s)\|_\infty = \left\| W_1(s)S_n(s)\left[I - W_5(s)\Delta(s)W_4(s)V_n(s)\right]^{-1} \right\|, \quad \forall \|\Delta(s)\|_\infty < 1$$

2) Robust stability

$$\|W_4(s)V_n(s)W_5(s)\|_\infty < 1$$

Result: (take W_5=1)
1) and 2) are both achieved if the controller is such that

$$\left\| \begin{bmatrix} W_1(s)S_n(s) \\ W_4(s)V_n(s) \end{bmatrix} \right\|_\infty < 1$$

The proof of the result follows from the following

Lemma

Let X(s) and Y(s) in L_∞ and Ψ a generic L_∞ function such that $\|Y_\infty\|<1$. If

$$\sup_\omega \left\{ \|X(j\omega)\| + \|Y(j\omega)\| \right\} < 1$$

Then

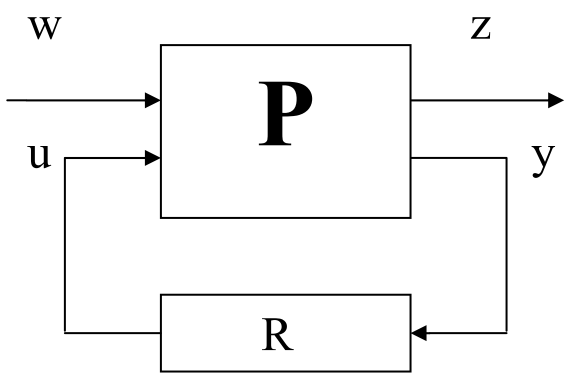
$$\sup_\omega \|X(j\omega)\| < 1$$
$$\sup_\omega \left\{ \left\| Y(j\omega)\left[I - \Psi(j\omega)X(j\omega)\right]^{-1} \right\| \right\} < 1$$

Proof of the Lemma

The first point directly follows from the assumption. The second point is trivial if X=0. Hence, assume X≠0 so that

$$\begin{aligned}
\left\| Y(j\omega)\left[I - \Psi(j\omega)X(j\omega)\right]^{-1} \right\| &\le \|Y(j\omega)\| \left\| \left[I - \Psi(j\omega)X(j\omega)\right]^{-1} \right\| \\
&= \frac{\|Y(j\omega)\|}{\sigma_{\min}(I - \Psi(j\omega)X(j\omega))} \le \frac{\|Y(j\omega)\|}{1 - \sigma_{\max}(\Psi(j\omega)X(j\omega))} \\
&\le \frac{\|Y(j\omega)\|}{1 - \sigma_{\max}(\Psi(j\omega))\sigma_{\max}(X(j\omega))} \le \frac{\|Y(j\omega)\|}{1 - \sigma_{\max}(X(j\omega))} \\
&\le \frac{\|Y(j\omega)\|}{1 - \|X(j\omega)\|} \le 1
\end{aligned}$$

The Standard Problem

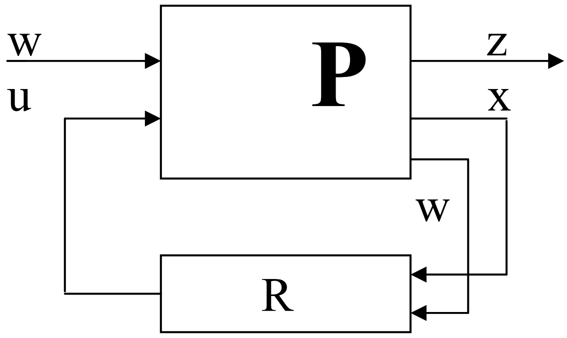


All the situations studied so far can be recast in the so-called standard problem: Find K(s) in such a way that:

- The closed-loop system is asymptotically stable
- $\|T(z,w,s)\|_{\infty} < \gamma$

Existence of a feasible R(s) and parametrization of all controllers such that the closed-loop norm between w and z be less than γ.

Full information



$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{12} u$$
$$y = \begin{bmatrix} x \\ w \end{bmatrix}$$

Assumptions
(A,B_2) stabilizable
$D_{12}'D_{12}>0$
(A_c,C_{1c}) detectable

$A_c = A-B_2(D_{12}'D_{12})^{-1} D_{12}'C_1$
$C_{1c} = (I-D_{12} (D_{12}'D_{12})^{-1} D_{12}')C_1$

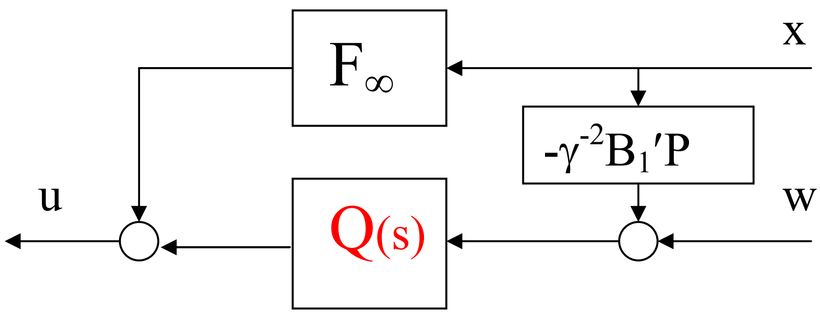
Solution of the Full Information Problem

Theorem

There exists a controller FI, feasible and such that $\|T(z,w,s)\|_\infty < \gamma$ if and only if there exists a positive semidefinite and stabilizing solution of the Riccati equation

$$A'P + PA - (PB_2 + C_1'D_{12})(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1) + P\frac{B_1B_1'}{\gamma^2}P + C_1'C_1 = 0$$

$$A - B_2(D_{12}'D_{12}')^{-1}(B_2'P + D_{12}'C_1) + \frac{B_1B_1'}{\gamma^2}P \quad \textit{stable}$$



$$F_\infty = -(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1)$$

Q(s) is a stable system, proper and satisfying $\|Q(s)\|_\infty < \gamma$.

Comments

- As $(\gamma \to \infty)$ the central controller (Q(s)=0) coincides with the H_2 optimal controller
- The parametrized family directly includes the only static controller $u=F_\infty x$.
- The proof of the previous result, in its general fashion, would require too much time. Indeed, the necessary part requires a digression on the Hankel-Toeplitz operator. If we drop off the result on the parametrization and limit ourselves to the static state-feedback problem, i.e. u=Kx, the following simple result holds:

Theorem

There exists a stabilizing control law u=Kx such that the norm of the system $(A+B_2K,B_1,C_1+D_{12}K)$ is less than γ if and only if there exists a positive definite solution of the inequality:

$$A'P + PA - (PB_2 + C'D_{12})(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1)$$
$$+ P\frac{B_1B_1'}{\gamma^2}P + C_1'C_1 < 0$$

Observation: LMI

$$P > 0$$
$$A'P + PA - (PB_2 + C'D_{12})(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1)$$
$$+ P\frac{B_1B_1'}{\gamma^2}P + C_1'C_1 < 0$$

$\Updownarrow$

Schur Lemma
$X = \gamma^2 P^{-1}$

$$X > 0$$
$$\begin{bmatrix} -XA_c' - A_cX - B_1B_1' + \gamma^2 B_2B_2' & XC_{1c}' \\ C_{1c}X & \gamma^2 I \end{bmatrix} > 0$$

$A_c = A-B_2(D_{12}'D_{12})^{-1} D_{12}'C_1$
$C_{1c} = (I-D_{12} (D_{12}'D_{12})^{-1} D_{12}')C_1$

Proof of Theorem

Assume that there exists K such that $A+B_2K$ is stable and the norm of the system $(A+B_2K,B_1,C_1+D_{12}K)$ is less than γ. Then, from the we know that there exists a positive definite solution of the inequality

$$\begin{aligned}&(A+B_2K)'P+P(A+B_2K)+\gamma^{-2}PB_1B_1'P+\\&+(C_1+D_{12}K)'(C_1+D_{12}K)<0\end{aligned} \qquad (*)$$

Now, defining

$$F=-(\mathrm{D}_{12}'\mathrm{D}_{12})^{-1}(B_2'P+\mathrm{D}_{12}'\mathrm{C}_1)$$

the Riccati inequality can be equivalently rewritten as

$$\begin{aligned}&A'P+PA-(PB_2+C'D_{12})(D_{12}'D_{12})^{-1}(B_2'P+D_{12}'C_1)+\\&P\frac{B_1B_1'}{\gamma^2}P+C_1'C_1+(K-F)'(K-F)<0\end{aligned}$$

so concluding the proof. Vice-versa, assume that there exists a positive definite solution of the inequality. Then, inequality (*) is satisfied with K=F, so that with such a K, the norm of the closed-loop transfer function is less than γ (BRL).

Parametrization of all algebraic "state-feedback" controllers

$$A_R = \begin{bmatrix} A & B_2 \end{bmatrix}$$
$$C_R = \begin{bmatrix} C_1 & D_{12} \end{bmatrix}$$
$$W = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$$

Define:

Theorem

The set of all controllers u=Kx such that the H_∞ norm of the closed-loop system is less than γ is given by:

$$K = W_2' W_1^{-1}$$
$$W_1 > 0$$
$$\begin{bmatrix} -A_R W - W A_R' - B_1 B_1' & W C_R' \\ C_R W' & \gamma^2 I \end{bmatrix} > 0$$

Proof of Theorem 7

We prove the theorem in the simple case in which $C_1' D_{12}=0$ and $D_{12}' D_{12}=I$. The LMI is equivalent to
If there exists W satisfying such an inequality, it follows that the inequality

$$P(A+B_2K)+(A+B_2K)'P+\gamma^{-2}PB_1B_1'P+C_1'C_1+KK'<0$$

$$K = W_2' W_1^{-1}$$
$$W_1 > 0$$
$$AW_1 + W_1A' + B_1B_1' + W_2B_2' + B_2W_2' + \gamma^{-2}W_1C_1'C_1W_1 + \gamma^{-2}W_2W_2' < 0$$

is satisfied with $K=W_2' W_1^{-1}$ and $P=\square^2 W_1^{-1}$. The result follows from the BRL. Vice-versa, assume that there exists P>0 satisfying this last inequality. The conclusion directly follows by letting $W_1=\gamma^2 P^{-1}$ e and $W_2=W_1 K'$.

Example

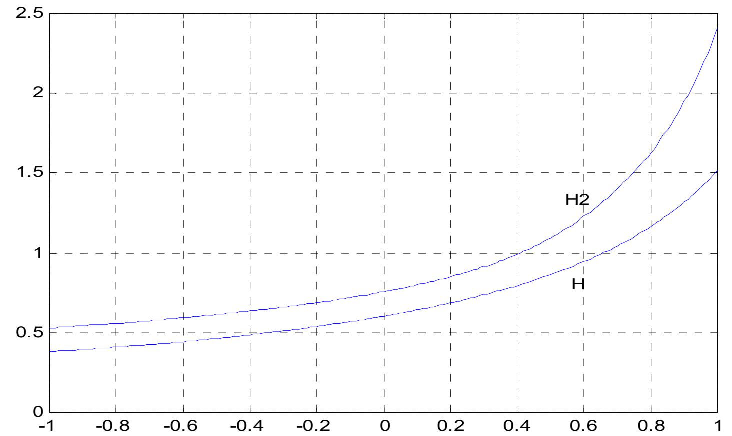
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (w+u)$$

$$z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

robustness $\Delta A = \begin{bmatrix} 0 & 0 \\ \Omega & 0 \end{bmatrix}$

$\gamma_{inf} = 0.75, \ \gamma = 0.76$

$\|T(z_1,w)\|_\infty$



Ω

Mixed Problem H_2/H_∞

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{12} u$$
$$\xi = Lx + Mu$$

The problem consists in finding u=Kx in such a way that

$$\min_K \left\| T(\xi, w, s, K) \right\|_2 : \left\| T(z, w, s, K) \right\|_\infty \leq \gamma$$

- For instance, take the case z=ξ. The problem makes sense since a blind minimization of the H_∞ infinity norm may bring to a serious deterioration of the H_2 (mean squares) performances.
- Obviously, the problem is non trivial only if the value of γ is included in the interval (γ_{inf} γ_2), where γ_{ing} is the infimun achievable H_∞ norm for T(z,w,s,K) as a function of K, whereas γ_2 is the H_∞ norm of T(z,w,s,K_{OTT}) where K_{OTT} is the optimal unconstrained H_2 controller.
- This problem has been tackled in different ways, but an analytic solution is not available yet. In the literature, many sub-optimal solutions can be found (Nash-game approach, convex optimization, inverse problem, …)

Post-Optimization procedure

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{12} u$$
$$\xi = Lx + Mu$$

simplifying assumptions $L'M=0$, $M'M=I$

Let K_{sub} a controller such that $\|T(z,w,s,K_{sub})\|_\infty<\gamma$. For $\alpha\in[0\ 2]$, define the matrices

$A_\alpha=A+(1-\alpha)^2B_2K_{sub}$

$B_{2\alpha}=(2\alpha-\alpha^2)^{1/2}B_2$

$C_\alpha=L+(1-\alpha)MK_{sub}$

and the standard Riccati equation

$$A_\alpha' P_\alpha + P_\alpha A_\alpha - P_\alpha B_{2\alpha} B_{2\alpha}' P_\alpha + C_\alpha' C_\alpha = 0 \quad (\alpha - RIC)$$

Notice that if (A,L) is detectable and (A,B_2) stabilizable, such an equation always admits the stabilizing solution (positive semidefinite) P_α for each $\alpha\in[0\ 2]$. For each $\alpha\in[0\ 2]$ we can write the family of controllers

$$\boxed{K_\alpha = (1-\alpha)K_{sub} - \alpha B_2' P_\alpha \qquad (\alpha - con)}$$

Post-Opt Algorithm

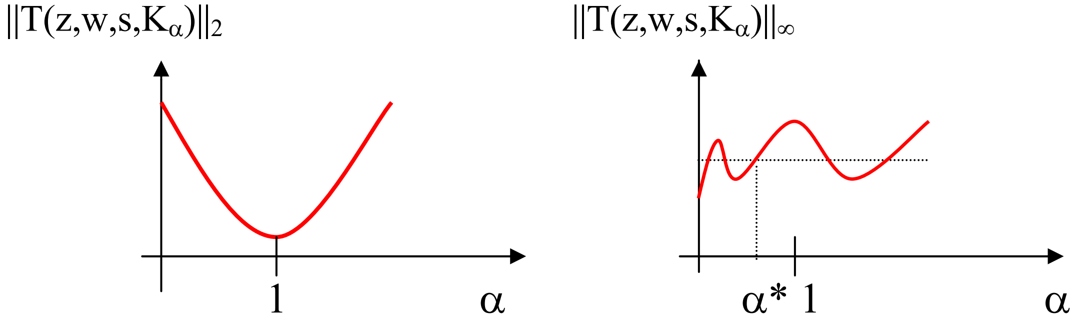
Theorem

Each controller K_α of the family (α-*con*) is a stabilizing controller and is such that

$$\|T(\xi, w, s, K_\alpha)\|_2 = \|T(\xi, w, s, K_{2-\alpha})\|_2$$

$$(1-\alpha)\frac{d}{d\alpha}\|T(\xi, w, s, K_\alpha)\|_2 \le 0$$

Interpretation: For $\alpha=0$, the controller is $K_0=K_{sub}$. Hence, $\|T(z,w,s,K_0)\|_\infty<\gamma$. For $\alpha=1$, the controller is $K_1=K_{OTT}$, so it coincides with the optimal unconstrained H_2 controller, hence $\|T(z,w,s,K_1)\|_\infty \ge \gamma$. One varies α till the value $\alpha*$ which is closest to $\alpha=1$ and such that $\|T(z,w,s,K_1)\|_\infty=\gamma$. The reason of the equality is in the fact (as it is possible to proof starting from the necessary conditions) that the optimal mixed controller K_{ottmix} satisfies $\|T(z,w,s,K_{ottmix})\|_\infty=\gamma$.



Proof of Theorem

First notice that the equation (α-RIC) canm be rewritten as:

$$(A+B_2K_\alpha)'P_\alpha + P_\alpha(A+B_2K_\alpha) + K_\alpha{}'K_\alpha + L'L = 0 \qquad (\alpha - RIC)$$

so that

$$\|T(\xi,w,s,K_\alpha)\|_2 = (\mathrm{Trace}(B_1{}'P_\alpha B_1))^{1/2}$$

The fact that the norm $\|T(\xi,w,s,K_\alpha)\|_2$ is symmetric with respect to $\alpha=1$ follows directly (α-RIC) by inspection. Taking the derivative with respect to α one obtains

$$F_\alpha{}'X_\alpha + X_\alpha F_\alpha - 2(1-\alpha)\Lambda_\alpha{}'\Lambda_\alpha = 0$$

where X_α is the derivate of P_α with respect to α and

$F_\alpha=A_\alpha-B_{2\alpha}B_{2\alpha}{}'P_\alpha$, $\quad \Lambda_\alpha=K_{sub}+P_\alpha B_2$

Matrix F_α is stable since P_α is the stabilizing solution of (α-RIC). Hence, for the Lyapunov Lemma the conclusion follows.

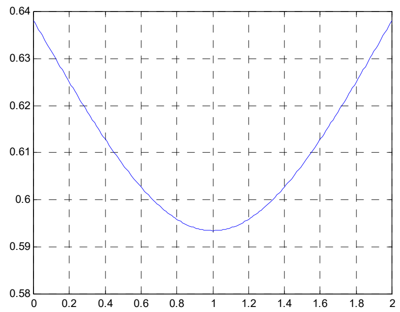
Example

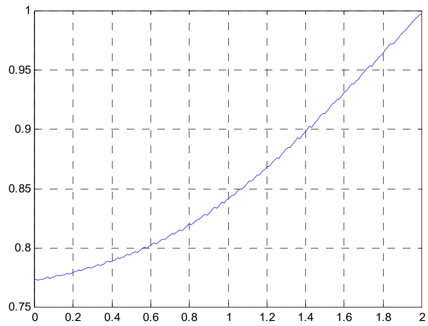
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (w+u)$$

$$z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \textit{robustness} \quad \Delta A = \begin{bmatrix} 0 & 0 \\ \Omega & 0 \end{bmatrix}$$

$\gamma_{inf} = 0.75$, $\gamma_2 = 0.8415$, $\gamma = 0.8$

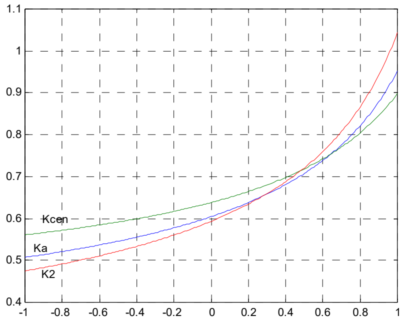
$K_{sub} = K_{cen} = [-0.6019 \quad -0.7676]$

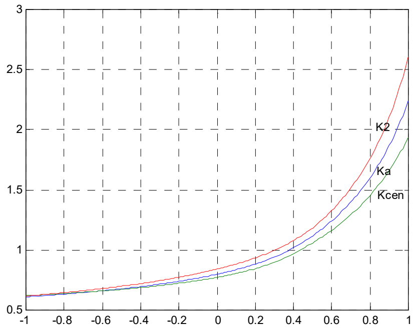




$\alpha^* = 0.56$, $K_{\alpha^*} = [-0.4981 \quad -0.5424]$

Performances with respect to Ω





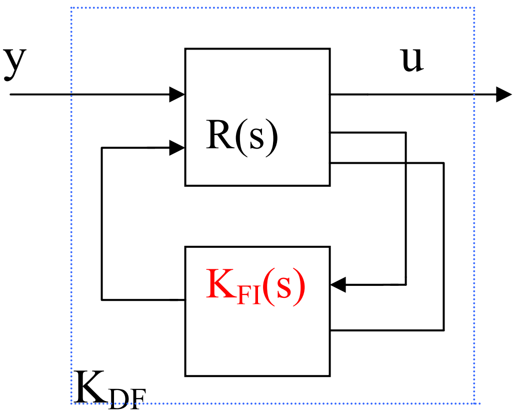
Disturbance Feedforward

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{12} u$$
$$y = C_2 x + w$$

Same assumptions as FI+stability of $A-B_1C_2$

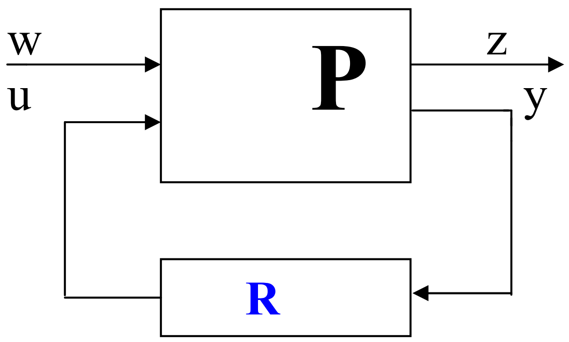
$C_2=0 \rightarrow$ direct compensation
$C_2 \neq 0 \rightarrow$ indirect compensation

$$R(s) = \left[\begin{array}{c|cc} A-B_1C_2 & B_1 & B_2 \\ \hline 0 & 0 & I \\ I & 0 & 0 \\ -C_2 & I & 0 \end{array}\right]$$



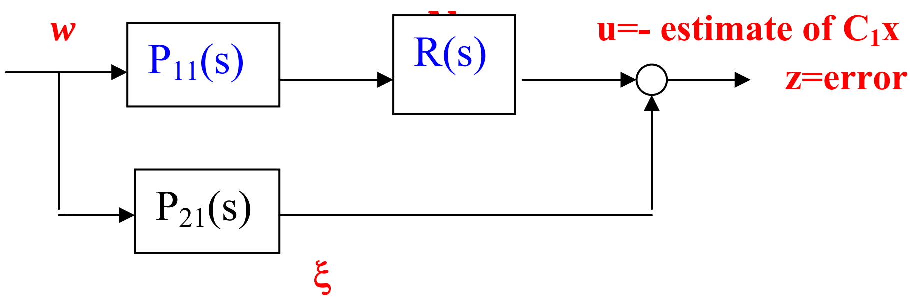
The proof of the DF theorem consist in verifying that the transfer function from w to z achieved by the FI regulator for the system A, B_1, B_2, C_1, D_{12} is the same as the transfer function from w to z obtained by using the regulator K_{DF} shown in the figure.

Output Estimation



$$\dot{x} = Ax + B_1 w + (B_2 u)$$
$$z = C_1 x + u$$
$$y = C_2 x + D_{21} w$$

$$u = -\hat{\xi}$$



Assumptions

(A, C_2) detectable

$D_{21}D_{21}' > 0$

(A_f, B_{1f}) stabilizable

$(A-B_2C_1)$ stable

$A_f = A - B_1 D_{21}'(D_{21}D_{21}')^{-1} C_2$

$B_{1f} = B_1(I - D_{21}' (D_{21}D_{21}')^{-1} D_{21})$

Solution of the
Output Estimation problem

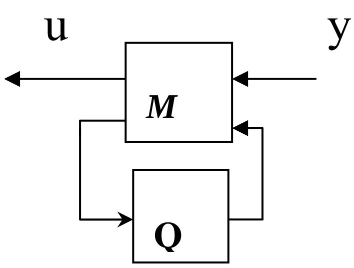
Theorem

There exists a feasible controllor (Filter) such that $\|T(z,w,s)\|_\infty < \gamma$ if and only if there exists the stabilizing positive semidefinite solution of the Riccati equation

$$A\Pi + \Pi A' - (\Pi C_2' + B_1 D_{21}')(D_{21}D_{21}')^{-1}(C_2\Pi + D_{21}B_1) + \Pi\frac{C_1'C_1}{\gamma^2}\Pi + B_1B_1' = 0$$

$$A - (\Pi C_2' + B_1 D_{21}')(D_{21}D_{21}')^{-1}C_2\Pi\frac{C_1'C_1}{\gamma^2} \quad \textit{stable}$$

M(s)=
$$\left[\begin{array}{c|cc} A_{ff} & L_\infty & -B_2\gamma^{-2}\Pi C_1' \\ \hline C_1 & 0 & I \\ C_{2f} & I_f & 0 \end{array}\right]$$



$$A_{ff} = A - \Pi C_2'(D_{21}D_{21}')^{-1}C_2 - B_2C_1$$

$$C_{2f} = (D_{21}D_{21}')^{-1}C_2$$

$$I_f = (D_{21}D_{21}')^{-1}$$

$$L_\infty = -(\Pi C_2' + B_1D_{21}')(D_{21}D_{21}')^{-1}$$

Q(s) is a proper stable system satisfying $\|Q(s)\|_\infty < \gamma$.

Proof of Theorem

It is enough to observe that the "transpose system"

$$\dot{\lambda} = A'\lambda + C'_1\varsigma + C_2'\nu$$

$$\mu = B_1'\lambda + D_{21}'\nu$$

$$\vartheta = B_2'\lambda + \varsigma$$

Has the same structure as the system defining the DF problem. Hence the solution is the "transpose" solution of the DF problem. It is worth noticing that the solution depends on the particular linear combination of the state that one wants to estimate.

Example

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1+\Omega \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_1$$

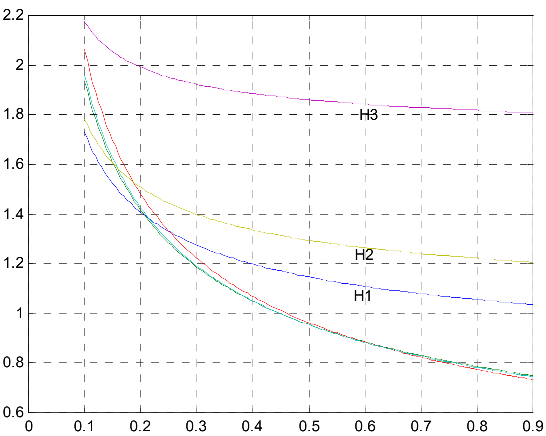
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + w_2$$

damping $\xi = (1-\Omega)/2$

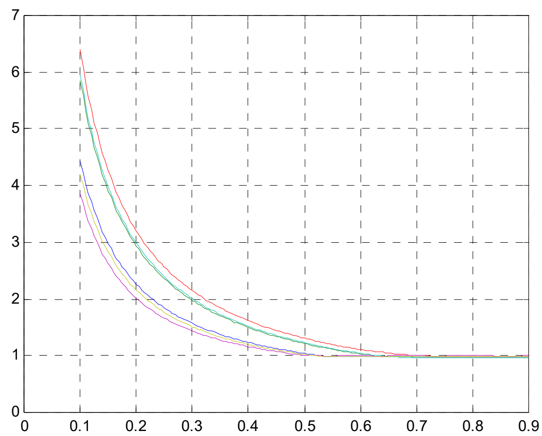
Let consider six filters:
Filter K1: Kalman filter. The noises are assumed to be white uncorrelated gaussian noises with identity intensities, namely W_1=1, W_2=1.
Filter K2: Kalman filter. The noises are assumed to be white uncorrelated gaussian noises with W_1=1, W_2=0.5.
Filter K3: Kalman filter. The noises are assumed to be white uncorrelated gaussian noises with W_1=0.5, W_2=1.
Filters H1,H2,H3: H_∞ filters with γ=1.1, γ=1.01, γ=1.005 (notice that with γ=1 the stabilizing solution of the Riccati equation does not exist).

Other techniques of Robust filtering are possible

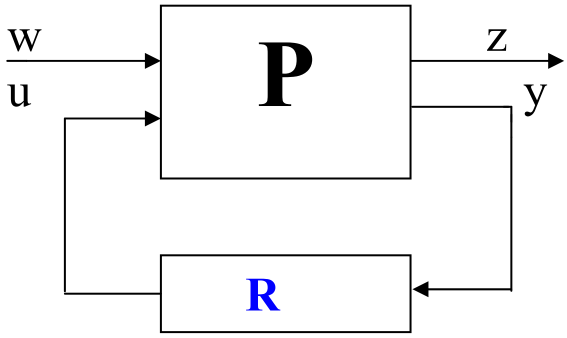
Norm H_2



Norm H_∞



Partial Information



$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{12} u$$
$$y = C_2 x + D_{21} w$$

Assumptions: FI + DF.

Theorem

There exists a feasible such that $\|T(z,w,s)\|_\infty < \gamma$ if and only if

- There exists the stabilizing positive semidefinite solution of

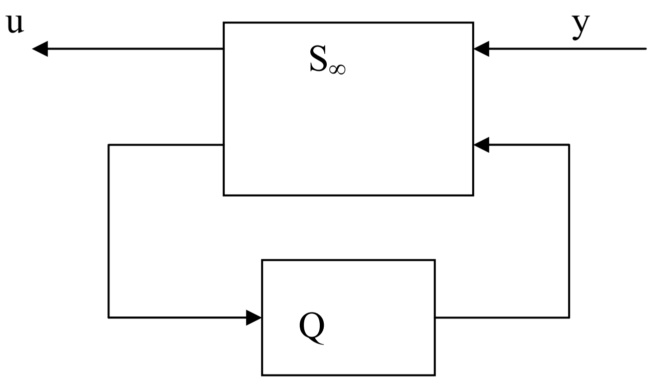
$$A'P + PA - (PB_2 + C_1'D_{12})(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1) + P\frac{B_1B_1'}{\gamma^2}P + C_1'C_1 = 0$$

- There exists the stabilizing positive semidefinite solution of

$$A\Pi + \Pi A' - (\Pi C_2' + B_1 D_{21}')(D_{21}D_{21}')^{-1}(C_2\Pi + D_{21}B_1') + \Pi\frac{C_1'C_1}{\gamma^2}\Pi + B_1B_1' = 0$$

- $\max_i \lambda_i(P\Pi) < \gamma^2$

Structure of the regulator



$$S_\infty(s) = \left[\begin{array}{c|cc} A_{fin} & B_{1fin} & B_{2fin} \\ \hline F_\infty & 0 & (D_{21}D_{21}')^{-1/2} \\ C_{2fin} & (D_{21}D_{21}')^{-1/2} & 0 \end{array}\right]$$

$$A_{fin} = A - B_2(D_{12}'D_{12})^{-1}D_{12}C_1 - B_2(D_{12}'D_{12})^{-1}B_2'P + \gamma^{-2}B_1B_1'P +$$
$$+ (I - \gamma^{-2}\Pi P)^{-1}L_\infty(C_2 + \gamma^{-2}D_{21}B_1'P)$$
$$B_{1fin} = -(I - \gamma^{-2}\Pi P)^{-1}L_\infty$$
$$B_{2fin} = (I - \gamma^{-2}\Pi P)^{-1}(B_2 + \gamma^{-2}\Pi C_1'D_{12})(D_{12}'D_{12})^{-1/2}$$
$$C_{2fin} = -(D_{21}D_{21}')^{-1}(C_2 + \gamma^{-2}D_{21}B_1'P)$$

Comments

It is easy to check that the central controller (Q(s)=0) is described by:

$$\dot{\xi} = A\xi + B_2 u + Z_\infty L_\infty (C_2\xi - y + D_{21}w^*) + B_1 w^*$$
$$u = F_\infty \xi$$

Where $Z_\infty=(I-\gamma^{-2}\Pi P)^{-1}$This closely resembles the structure of the optimal H_2 controller. Notice, however, that the well-known separation principle does not hold for the presence of the worst disturbance $w^*=\gamma^{-2}B_1'P\xi$.

Proof of Theorem (sketch)
Define the variables *r* e *q* as:
$w = r + \gamma^{-2}B_1'Px$
$q = u - F_\infty x$
In this way, the system becomes

$$\dot{x} = (A + \gamma^{-2}B_1B_1'P)x + B_1 r + B_2 u$$
$$q = -F_\infty x + u$$
$$y = (C_2 + \gamma^{-2}D_{21}B_1')x + D_{21}w$$

This system has the same structure as the one of the OE problem. Hence, the solution can be obtained from the solution of the OE problem, by recognizing that the solution Π_t of the relevant Riccati equation is related to the solutions P e Π above in the following way:

$$\Pi_t = \Pi(I - \gamma^{-2}P\Pi)^{-1}$$