

# Notes of robust control

- Zeros (continuous-time systems)
- Parametrization (continuous-time systems)
- $H_2$  control (continuous-time systems)
- $H_\infty$  control (continuous-time systems)
- $H_\infty$  filtering in discrete-time (discrete-time systems)

# ZEROS AND POLES OF MULTIVARIABLE SYSTEMS

## SUMMARY

- Transmission zeros

Rank property and time-domain characterization

- Invariant zeros

Rank property and time-domain characterization

- Decoupling zeros

- Infinite zero structure

## (Transmission) Zeros and poles of a rational matrix

$$G(s) = C(sI - A)^{-1}B + D, \quad p \text{ outputs}, \quad m \text{ inputs}$$

Basic assumption:

$$\text{normal rank}[G(s)] = \min(p, m) = r$$

Smith-McMillan form of  $G(s)$

$$M(s) = \begin{bmatrix} f_1(s) & 0 & \cdots & 0 & 0 \\ 0 & f_2(s) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f_r(s) & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$f_i(s) = \frac{\mathbf{e}_i(s)}{\mathbf{y}_i(s)}, \quad i = 1, 2, \dots, r$$

$\mathbf{e}_i$  divides  $\mathbf{e}_{i+1}$

$\mathbf{y}_{i+1}$  divides  $\mathbf{y}_i$

- The transmission zeros are the roots of  $\mathbf{e}_i(s)$ ,  $i=1,2,\dots,r$
- The (transmission) poles are the roots of  $\mathbf{y}_i(s)$ ,  $i=1,2,\dots,r$

## Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -10 & 7 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} -13 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence

$$G(s) = \begin{bmatrix} (s-3)(s+1) \\ (s-3)(s-2) \end{bmatrix} \frac{1}{(s-2)(s-5)}$$

$$r=1, \quad M(s) = \begin{bmatrix} s-3 \\ 0 \end{bmatrix} \frac{1}{(s-2)(s-5)}$$

There is one transmission zero at  $s=3$  and two poles in  $s=2$  and  $s=5$ .

# Rank property of transmission zeros

## Theorem 1

The complex number  $\lambda$  is a transmission zero of  $G(s)$  iff there exists polynomial vector  $z(s)$  such that  $z(\lambda) \neq 0$  and

$$\begin{cases} \lim_{s \rightarrow \lambda} G(s)z(s) = 0 & \text{if } p \geq m \\ \lim_{s \rightarrow \lambda} G(s)'z(s) = 0 & \text{if } p \leq m \end{cases}$$

## Remark

In the square case ( $p=m$ ), it follows

$$\det [G(s)] = \frac{\prod_{i=1}^r e_i(s)}{\prod_{i=1}^r y_i(s)}$$

so that it is possible to catch all zeros and poles if and only if there are not cancellations.

### Proof of the theorem

Assume that  $p \geq m$  and let  $M(s)$  be the Smith-Mc Millan form of  $G(s)$ , i.e.  $G(s)=L(s)M(s)R(s)$  for suitable polynomial and unimodular matrices  $L(s)$  (of dimension  $p$ ) and  $R(s)$  (of dimensions  $m$ ). It is possible to write:

$$M(s) = \begin{bmatrix} E(s)\Psi^{-1}(s) \\ 0 \end{bmatrix}$$

where  $E(s)=\text{diag}\{\epsilon_i(s)\}$  and  $\Psi(s)=\text{diag}\{\psi_i(s)\}$ . Now assume that  $\lambda$  is a zero of the polynomial  $e_k(s)$ . Recall that  $R(s)$  is unimodular and take the polynomial vector  $z(s)=[R^{-1}(s)]_k$ , i.e. the  $k$ -th column of the inverse of  $R(s)$ . Since  $\psi_k(\lambda) \neq 0$ , it follows that  $G(s)z(s)=L(s)E(s)\Psi^{-1}(s)R(s)z(s)=L(s)E(s) [\Psi^{-1}(s)]_k = L(s)[E(s)]_k/\psi_k(s)=[L(s)]_k \epsilon_k(s)/\psi_k(s)$ . This quantity goes to zero as  $s$  tends to  $\lambda$ . Vice-versa, if  $z(s)$  is such that  $G(s)z(s)$  goes to zero as  $s$  tends to  $\lambda$ , then the limit as  $s$  tends to  $\lambda$  of  $E(\lambda)Y^{-1}(s)R(\lambda)z(\lambda)$  being zero means that at least one polynomial  $\epsilon_k(s)$  should vanish at  $s=\lambda$ .

The proof in the opposite case ( $p \leq m$ ) follows the same lines and therefore is omitted.

## Example

$$G(s) = \begin{bmatrix} 1 & (s-1)(s+2) \\ 0 & (s-1)^2 \end{bmatrix} \frac{1}{(s-1)(s+2)}$$

Hence,

$$G(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{(s-1)(s+2)} & 0 \\ 0 & \frac{s-1}{s+2} \end{bmatrix} \frac{1}{(s-1)(s+2)}$$

so that, with

$$z(s) = \begin{bmatrix} -(s-1)(s+2) \\ 1 \end{bmatrix}$$

it follows

$$\lim_{s \rightarrow 1} G(s)z(s) = 0$$

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Notice that it is not true that  $G(s)z(1)$  tends to zero as  $s$  tends to one. .

# Time-domain characterization of transmission zeros and poles

Let  $\Sigma=(A,B, C ,D)$  be the state-space system with transfer function  $G(s)=C(sI-A)^{-1}B+D$ . Let  $\Sigma'=(A',C',B',D')$  the “transpose” system whose transfer function is  $G(s)'=B'(sI-A')^{-1}C'+D'$ . Now, let

$$\hat{\Sigma} = \begin{cases} \Sigma & p \geq m \\ \Sigma' & p \leq m \end{cases} \quad \hat{G}(s) = \begin{cases} G(s) & p \geq m \\ G(s)' & p \leq m \end{cases}$$

Finally, let  $\delta^i(t)$  denote the i-th derivative of the impulsive “function”.

## Theorem 2

(a) The complex number  $\lambda$  is a pole of  $\hat{\Sigma}$  iff there exists an impulsive input

$$u(t) = \sum_{i=0}^n \mathbf{a}_i \mathbf{d}^i(t)$$

such that the forced output  $y_f(t)$  of  $\hat{\Sigma}$  is

$$y_f(t) = y_0 e^{\lambda t}, \quad \forall t \geq 0$$

(b) The complex number  $\lambda$  is a zero of  $\hat{\Sigma}$  iff there exists an exponential/impulsive input

$$u(t) = u_0 e^{\lambda t} + \sum_{i=0}^n \mathbf{a}_i \mathbf{d}^i(t), \quad u_0 \neq 0$$

such that the forced output  $y_f(t)$  of  $\Sigma$  is

$$y_f(t) = 0, \quad \forall t \geq 0$$

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Zeros blocking property. The proof of the theorem is not reported here for space limitations.

## Invariant zeros

Given a system (A,B,C,D) with transfer function  $G(s)=C(sI-A)^{-1}B+D$ , the polynomial matrix

$$P(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$$

is called the **system matrix**.

Consider the Smith form  $S(s)$  of  $P(s)$ :

$$S(s) = \begin{bmatrix} \mathbf{a}_1(s) & 0 & \dots & 0 & 0 \\ 0 & \mathbf{a}_2(s) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{a}_n(s) & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$\mathbf{a}_i$  divides  $\mathbf{a}_{i+1}$ .

The **invariant zeros** are the roots of  $\alpha_i(s)$ ,  $i=1,2,\dots,v$

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Notice that

$$P(s) = \begin{bmatrix} sI - A & 0 \\ C & I \end{bmatrix} \begin{bmatrix} (sI - A)^{-1} & 0 \\ 0 & G(s) \end{bmatrix} \begin{bmatrix} sI - A & -B \\ 0 & I \end{bmatrix}$$

so that the rank of  $P(s)$  is  $\varpi=n+\text{rank}(G(s))$



# Rank property of invariant zeros

## Theorem 3

The complex number  $\lambda$  is an invariant zero of  $(A,B,C,D)$  iff  $P(s)$  loses rank in  $s=\lambda$ , i.e. iff there exists a nonzero vector  $z$  such that

$$\begin{cases} P(\lambda)z = 0 & \text{if } p \geq m \\ P(\lambda)'z = 0 & \text{if } p \leq m \end{cases}$$

Notice that if  $T$  is the transformation for a change of basis in the state-space, then

$$\bar{P}(s) = \begin{bmatrix} sI - \bar{A} & -\bar{B} \\ \bar{C} & D \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}$$

Therefore, the invariant zeros are invariant with respect to a change of basis.

### Proof of Theorem 3

Consider the case  $p \geq m$  and let  $P(s) = L(s)S(s)R(s)$ , where  $R(s)$  and  $L(s)$  are suitable polynomial unimodular matrices. Hence, if  $P(\lambda)z=0$  for some  $z \neq 0$ , then  $S(\lambda)R(\lambda)z=0$  so that  $S(\lambda)y=0$ , with  $y \neq 0$ . Hence,  $\lambda$  must be the root of at least one polynomial  $\alpha_i(s)$ . Vice-versa, if  $\alpha_k(\lambda)$  for some  $k$ , then  $z = [R^{-1}(\lambda)]_k$  is such that  $P(\lambda)z = L(\lambda)S(\lambda)R(\lambda)z = L(\lambda)S(\lambda)R(\lambda)[R^{-1}(\lambda)]_k = L(\lambda)[S(\lambda)]_k = 0$ .

## Time-domain characterization of invariant zeros

Let  $\Sigma=(A,B, C ,D)$  be the state-space system with transfer function  $G(s)=C(sI-A)^{-1}B+D$ . Let  $\Sigma'=(A',C',B',D')$  the “transpose” system whose transfer function is  $G(s)'=B'(sI-A')^{-1}C'+D'$ . Now, let

$$\hat{\Sigma} = \begin{cases} \Sigma & p \geq m \\ \Sigma' & p \leq m \end{cases} \quad \hat{G}(s) = \begin{cases} G(s) & p \geq m \\ G(s)' & p \leq m \end{cases}$$

### Theorem 4

The complex number  $\lambda$  is an invariant zero  $\Sigma$  iff at least one of the two following conditions holds:

- (i)  $\lambda$  is an eigenvalue of the unobservable part of  $\hat{\Sigma}$
- (ii) there exist two vectors  $x_0$  and  $u_0 \neq 0$  such that the forced output of  $\hat{\Sigma}$  corresponding to the input  $u(t)=u_0 e^{\lambda t}$ ,  $t \geq 0$  and the initial state  $x(0)=x_0$  is identically zero for  $t \geq 0$ .

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Zeros blocking property. The proof of the theorem is not reported here for space limitations.

# Transmission vs invariant zeros

## Theorem 5

- (i) A transmission zero is also an invariant zero.
- (ii) Invariant and transmission zeros of a system in minimal form do coincide.

**Proof of point (i).** Let  $p \geq m$  and let  $\lambda$  be a transmission zero. Then, there exists an exponential/impulsive input

$$u(t) = u_0 e^{It} + \sum_{i=0}^n \mathbf{a}_i \mathbf{d}^{(i)}(t) \text{ such that the forced output is zero, i.e.}$$

$$y_f(t) = C \int_0^t e^{A(t-\tau)} B \left[ u_0 e^{I\tau} + \sum_{i=0}^n \mathbf{a}_i \mathbf{d}^{(i)}(\tau) \right] d\tau + D u_0 e^{It} = 0, t > 0. \text{ Letting}$$

$$x_0 = \begin{bmatrix} B & AB & \dots & A^n B \end{bmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}, \text{ it follows}$$

$$C e^{At} x_0 + C \int_0^t e^{A(t-\tau)} B u_0 e^{I\tau} d\tau + D u_0 e^{It} = 0, t > 0$$

so that  $\lambda$  is an invariant zero. The proof in the case  $p \leq m$  is the same, considering the transpose system.

**Proof of point (ii).** Let  $p \geq m$ . We have already proven that a transmission zero is an invariant zero. Then, consider a minimum system and an invariant zero  $\lambda$ . Hence  $A\mathbf{v} + B\mathbf{w} = \lambda\mathbf{w}$ ,  $C\mathbf{v} + D\mathbf{w} = 0$ . Notice that  $\mathbf{w} \neq 0$  since the system is observable. Moreover, thanks to the reachability condition there exists a vector

$$\mathbf{v} = \begin{bmatrix} B & AB & \dots & A^n B \end{bmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}$$

Since  $\lambda$  is an invariant zero there exists an initial state  $\mathbf{x}(0) = \mathbf{v}$  and an input  $u(t) = \mathbf{w} e^{\lambda t}$ , such that

$$C e^{At} \mathbf{v} + C \int_0^t e^{A(t-\tau)} B \mathbf{w} e^{I\tau} d\tau + D \mathbf{w} e^{It} = 0, t > 0$$

By noticing that  $C e^{At} \mathbf{v} = C e^{At} \sum_{i=0}^n A^i B \mathbf{a}_i = \int_0^t C e^{A(t-\tau)} B \sum_{i=0}^n \mathbf{a}_i \mathbf{d}^{(i)}(\tau) d\tau$ , it follows,

$$C \int_0^t e^{A(t-\tau)} B \left[ u_0 e^{I\tau} + \sum_{i=0}^n \mathbf{a}_i \mathbf{d}^{(i)}(\tau) \right] d\tau + D u_0 e^{It} = 0, t > 0, \text{ which means that } \lambda \text{ is a transmission zero.}$$

## Decoupling zeros

$$P_C(s) = \begin{bmatrix} sI - A \\ C \end{bmatrix} \quad P_B(s) = [sI - A \quad -B]$$

### Associated Smith forms

$$S_C(s) = \begin{bmatrix} \text{diag} \{ \mathbf{a}_i^C(s) \} \\ 0 \end{bmatrix} \quad S_B(s) = [\text{diag} \{ \mathbf{a}_i^B(s) \} \quad 0]$$

The output decoupling zeros are the roots of  $\alpha_1^C(s) \alpha_2^C(s) \dots \alpha_n^C(s)$ , the input decoupling zeros are the roots of  $\alpha_1^B(s) \alpha_2^B(s) \dots \alpha_n^B(s)$  and the input-output decoupling zeros are the common roots of both.

### Lemma

- (i) A complex number  $\lambda$  is an output decoupling zero iff there exists  $z \neq 0$  such that  $P_C(\lambda)z = 0$ .
- (ii) A complex number  $\lambda$  is an input decoupling zero iff there exists  $w \neq 0$  such that  $P_B(\lambda)'w = 0$ .
- (iii) A complex number  $\lambda$  is an input-output decoupling zero iff there exist  $z \neq 0$  and  $w \neq 0$  such that  $P_C(\lambda)z = 0$ ,  $P_B(\lambda)'w = 0$ .
- (iv) For square systems the set of decoupling zeros is a subset of the set of invariant zeros.
- (v) For square invertible systems the transmission zeros of the inverse system coincide with the poles of the system and the invariant zeros of the inverse system coincide with the eigenvalues of the system.

# Infinite zero structure

## Mc Millan form over the ring of causal rational matrices

Given the transfer function  $G(s)$  there exists two unimodular matrices in that ring (bi-proper rational functions) such that

$$G(s) = V(s) \begin{bmatrix} M(s) & 0 \\ 0 & 0 \end{bmatrix} W(s), \quad M(s) = \text{diag}\{s^{-d_1}, s^{-d_2}, \dots, s^{-d_r}\}$$

where  $\delta_1, \delta_2, \dots, \delta_r$  are integers such that  $\delta_1 \leq \delta_2 \leq \dots \leq \delta_r$ . They describes the structure of the infinity zeros.

- Inversion
- Model matching
- .....

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Example: compute the finite and infinite zeros of:

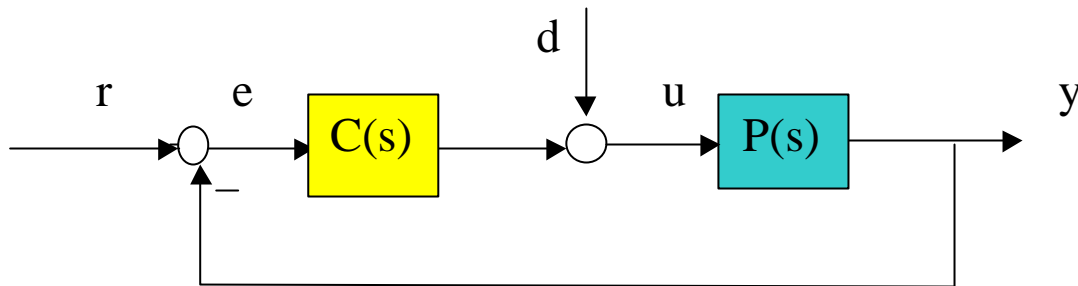
$$G(s) = \begin{bmatrix} \frac{s+1}{s-1} & 0 & \frac{s+1}{s-1} \\ 1 & \frac{1}{s} & \frac{s+1}{s} \\ \frac{1}{s} & 0 & \frac{s^3 - 3s^2 + s + 5}{s^3(s-3)} \end{bmatrix}$$

# PARAMETRIZATION OF STABILIZING CONTROLLERS

## Summary

- BIBO stability and internal stability of feedback systems
- Stabilization of SISO systems. Parameterization and characterization of stabilizing controllers
- Coprime factorization
- Stabilization of MIMO systems. Parameterization and characterization of stabilizing controllers
- Strong stabilization

## Stability of feedback systems



The problem consists in the analysis of the asymptotic stability (internal stability) of closed-loop system from some characteristic transfer functions. This result is useful also for the design.

### THEOREM 1

The closed loop system is asymptotically stable if and only if the transfer matrix  $G(s)$  from the input to the output

$$\begin{bmatrix} r \\ d \end{bmatrix} \xrightarrow{G(s)} \begin{bmatrix} e \\ u \end{bmatrix}$$

is stable.

$$G(s) = \begin{bmatrix} (I + P(s)C(s))^{-1} & -(I + P(s)C(s))^{-1}P(s) \\ (I + C(s)P(s))^{-1}C(s) & (I + C(s)P(s))^{-1} \end{bmatrix}$$

## Stability of interconnected SISO systems

The closed loop system is asymptotically stable if and only if the three transfer functions

$$S(s) = \frac{1}{1 + C(s)P(s)}, R(s) = \frac{C(s)}{1 + C(s)P(s)}, H(s) = \frac{P(s)}{1 + C(s)P(s)}$$

are stable.

- Notice that if  $S(s)$ ,  $R(s)$ ,  $H(s)$  are stable, then also the *complementary sensitivity*

$$T(s) = \frac{C(s)P(s)}{1 + C(s)P(s)} = 1 - S(s)$$

is stable.

- The stability of the three transfer functions are made to avoid prohibited (unstable) cancellations between  $C(s)$  and  $P(s)$ .

### Example

Let  $C(s) = (s-1)/(s+1)$ ,  $P(s) = 1/(s-1)$ . Then  $S(s) = (s+1)/(s+2)$ ,  
 $R(s) = (s-1)/(s+2)$  e  $H(s) = (s+1)/(s^2+s-2)$



## Comments

The proof of Theorem 1 can be carried out pursuing different paths. A simple way is as follows. Let assume that  $C(s)$  and  $P(s)$  are described by means of *minimal realizations*  $C(s)=(A,B,C,D)$ ,  $P(s)=(F,G,H,E)$ . It is possible to write a realization of the closed-loop system as  $\Sigma_{cl}=(A_{cl},B_{cl},C_{cl},D_{cl})$ . Obviously, if  $A_{cl}$  is asymptotically stable, then  $G(s)$  is stable. Vice-versa, if  $G(s)$  is stable, asymptotic stability of  $A_{cl}$  follows from being  $\Sigma_{cl}=(A_{cl},B_{cl},C_{cl},D_{cl})$  a minimal realization. This can be seen through the well known PBH test.

In our example, it follows

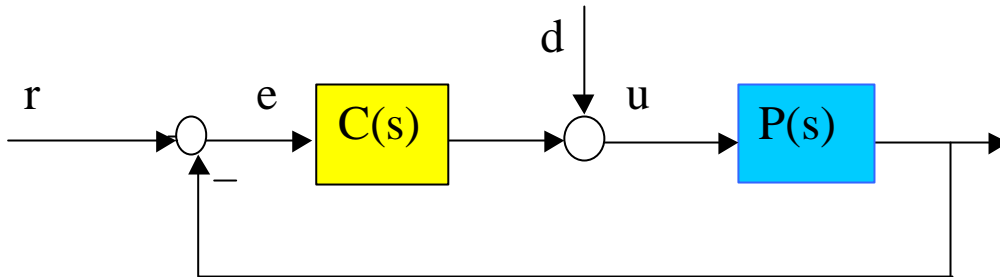
$$A_{cl} = \begin{bmatrix} 0 & -2 \\ -1 & -1 \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad C_{cl} = \begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix}, \quad D_{cl} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} \frac{s+1}{s+2} & \frac{-(s+1)}{(s-1)(s+2)} \\ \frac{s-1}{s+2} & \frac{s+1}{s+2} \end{bmatrix} \xrightarrow[SMM]{\triangleright} \begin{bmatrix} \frac{1}{(s-1)(s+2)} & 0 \\ 0 & (s+1)(s-1) \end{bmatrix}$$

Hence,  $A_{cl}$  is unstable and the closed-loop system is reachable and observable. This means that  $G(s)$  is unstable as well.

# Parametrization of stabilizing controllers

## 1° case: SISO systems and $P(s)$ stable



### THEOREM 2

The family of all controllers  $C(s)$  such that the closed-loop system is asymptotically stable is:

$$C(s) = \frac{Q(s)}{1 - P(s)Q(s)}$$

where  $Q(s)$  is proper and stable and  $Q(\infty)P(\infty) \neq 1$ .

## Proof of Theorem 2

Let  $C(s)$  be a stabilizing controller and define  $Q(s) = C(s)/(1+C(s)P(s))$ . Notice that  $Q(s)$  is stable since it is the transfer function from  $r$  to  $u$ . Hence  $C(s)$  can be written as  $C(s) = Q(s)/(1-P(s)Q(s))$ , with  $Q(s)$  stable and, obviously, the stability of both  $Q(s)$  and  $P(s)$  implies that  $Q(\infty)P(\infty) = C(\infty)/(1+C(\infty)P(\infty)) \neq 1$ .

Vice-versa, assume that  $Q(s)$  is stable and  $Q(\infty)P(\infty) \neq 1$ . Define  $C(s) = Q(s)/(1-P(s)Q(s))$ . It results that

$$\begin{aligned} S(s) &= 1/(1+C(s)P(s)) = 1-P(s)Q(s), \\ R(s) &= C(s)/(1+C(s)P(s)) = Q(s), \\ H(s) &= P(s)/(1+C(s)P(s)) = P(s)(1-P(s)Q(s)) \end{aligned}$$

are stable. This means that the closed-loop system is asymptotically stable.

# Parametrization of stabilizing controllers

## 2° case: SISO systems and generic $P(s)$

It is always possible to write (**coprime factorization**)

$$P(s) = \frac{N(s)}{M(s)}$$

where  $N(s)$  and  $M(s)$  are stable rational coprime functions, i.e. such that there exist two stable rational functions  $X(s)$  e  $Y(s)$  satisfying (**equation of Diofanto, Aryabatta, Bezout**)

$$N(s)X(s) + M(s)Y(s) = 1$$

### **THEOREM 3**

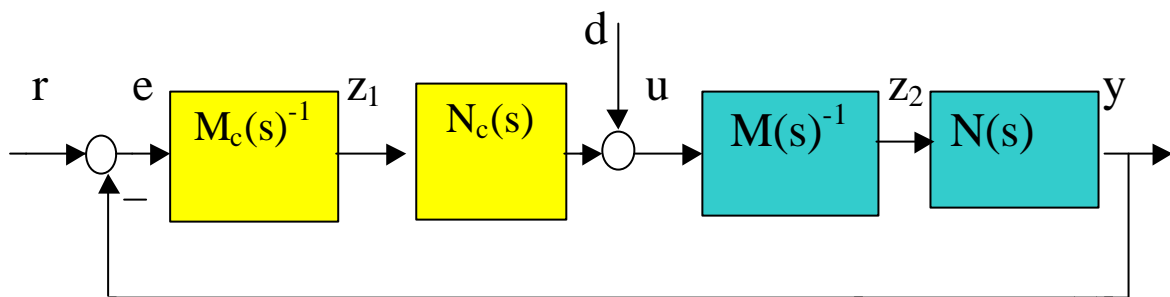
The family of all controllers  $C(s)$  such that the closed-loop system is well-posed and asymptotically stable is:

$$C(s) = \frac{X(s) + M(s)Q(s)}{Y(s) - N(s)Q(s)}$$

where  $Q(s)$  is proper, stable and such that  $Q(\infty)N(\infty) \neq Y(\infty)$ .

## Comments

The proof of the previous theorem can be carried out following different ways. However, it requires a preliminary discussion on the concept of coprime factorization and on the stability of factorized interconnected systems.



### LEMMA 1

Let  $P(s)=N(s)/M(s)$  e  $C(s)=N_c(s)/M_c(s)$  stable coprime factorizations. Then the closed-loop system is asymptotically stable if and only if the transfer matrix  $K(s)$  from  $[r' d']'$  to  $[z_1' z_2']'$  is stable.

$$\begin{bmatrix} r \\ d \end{bmatrix} \xrightarrow{K(s)} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$K(s) = \begin{bmatrix} M_c(s) & N(s) \\ -N_c(s) & M(s) \end{bmatrix}^{-1}$$

## Proof of Lemma 1

Define four stable functions  $X(s), Y(s), X_c(s), Y_c(s)$  such that

$$\begin{aligned}X(s)N(s) + Y(s)M(s) &= 1 \\X_c(s)N_c(s) + Y_c(s)M_c(s) &= 1\end{aligned}$$

To proof the Lemma it is enough to resort to Theorem 1, by noticing that the transfer functions  $K(s)$  and  $G(s)$  (from  $[r' \ d']'$  to  $[e' \ u']'$ ) are related as follows:

$$K(s) = \begin{bmatrix} Y_c(s) & X_c(s) \\ -X(s) & Y(s) \end{bmatrix} G(s) - \begin{bmatrix} 0 & X_c \\ -X & 0 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & -N \\ N_c & 0 \end{bmatrix} K(s)$$

### Proof of Theorem 3

Assume that  $Q(s)$  is stable and  $Q(\infty)N(\infty) \neq Y(\infty)$ . Moreover, define  $C(s) = (X(s) + M(s)Q(s)) / (Y(s) - N(s)Q(s))$ . It follows that

$$1 = N(s)X(s) + M(s)Y(s) = N(s)(X(s) + M(s)Q(s)) + M(s)(Y(s) - N(s)Q(s))$$

so that the functions  $X(s) + M(s)Q(s)$  e  $Y(s) - N(s)Q(s)$  defining  $C(s)$  are coprime (besides being both stable). Hence, the three characteristic transfer functions are:

$$\begin{aligned} S(s) &= 1 / (1 + C(s)P(s)) = M(s)(Y(s) - N(s)Q(s)), \\ R(s) &= C(s) / (1 + C(s)P(s)) = M(s)(X(s) + M(s)Q(s)), \\ H(s) &= P(s) / (1 + C(s)P(s)) = N(s)(Y(s) - N(s)Q(s)) \end{aligned}$$

Since they are all stable, the closed-loop system is asymptotically stable as well (Theorem 1).

Vice-versa, assume that  $C(s) = N_c(s) / M_c(s)$  (stable coprime factorization) is such that the closed-loop system is well-posed and asymptotically stable. Define  $Q(s) = (Y(s)N_c(s) - X(s)M_c(s)) / (M(s)M_c(s) + N(s)N_c(s))$ . Since the closed-loop system is asymptotically stable and  $(N_c, M_c)$  are coprime, then, in view of Lemma 1 it follows that

$$\begin{aligned} Q(s) &= (Y(s)N_c(s) - X(s)M_c(s)) / (M(s)M_c(s) + N(s)N_c(s)) = \\ &= [0 \ I]K(s)[Y(s)' - X(s)']' \end{aligned}$$

is stable. This leads to

$$C(s) = (X(s) + M(s)Q(s)) / (Y(s) - N(s)Q(s)).$$

Finally,  $Q(\infty)N(\infty) \neq Y(\infty)$ , as can be easily be verified.

# Coprime Factorization

## 1° case: SISO systems

Lemma 1 makes reference to a factorized description of a transfer function. Indeed, it is easy to see that it is always possible to write a transfer function as the ratio of two stable transfer functions **without common divisors** (coprime). For example

$$G(s) = \frac{s + 1}{(s - 1)(s + 10)(s - 2)} = N(s)M(s)^{-1}$$

with

$$N(s) = \frac{1}{(s + 10)(s + 1)}, \quad M(s) = \frac{(s - 1)(s - 2)}{(s + 1)^2}$$

It results

$$N(s)X(s) + M(s)X(s) = 1$$

with

$$X(s) = \frac{4(16s - 5)}{(s + 1)}, \quad Y(s) = \frac{(s + 15)}{(s + 10)}$$

Euclide's Algorithm



# Coprime Factorization

## 1° case: MIMO systems

In the MIMO case, we need to distinguish between right and left factorizations.

### Right and left factorization

Given  $P(s)$ , find four proper and stable transfer matrices such that:

$$P(s) = N_d(s)M_d(s)^{-1} = M_s(s)^{-1}N_s(s)$$

$$N_d(s), M_d(s) \text{ right coprime } X_d(s)N_d(s) + Y_d(s)M_d(s) = I$$

$$N_s(s), M_s(s) \text{ left coprime } N_s(s)X_s(s) + M_s(s)Y_s(s) = I$$

### Double coprime factorization

Given  $P(s)$ , find eight proper and stable transfer functions such that:

$$(i) \quad P(s) = N_d(s)M_d(s)^{-1} = M_s(s)^{-1}N_s(s)$$

$$(ii) \quad \begin{bmatrix} Y_d(s) & X_d(s) \\ -N_s(s) & M_s(s) \end{bmatrix} \begin{bmatrix} M_d(s) & -X_s(s) \\ N_d(s) & Y_s(s) \end{bmatrix} = I$$

Hermite's algorithm

Observer and control law

# Construction of a stable double coprime factorization

Let  $(A,B,C,D)$  a stabilizable and detectable realization of  $P(s)$  and conventionally write  $P=(A,B,C,D)$  .

## THEOREM 4

Let  $F$  e  $L$  two matrices such that  $A+BF$  e  $A+LC$  are stable. Then there exists a stable double coprime factorization, given by:

$$M_d=(A+BF, B, F, I), \quad N_d=(A+BF, B, C+DF, D)$$

$$Y_s=(A+BF, L, -C-DF, I), \quad X_s=(A+BF, L, F, 0)$$

$$M_s=(A+LC, L, C, I), \quad N_s=(A+LC, B+LD, C, D)$$

$$Y_d=(A+LC, B+LD, -F, I), \quad X_d=(A+LC, L, F, 0)$$

### Proof of Theorem 4

The existence of stabilizing  $F$  e  $L$  is ensured by the assumptions of stabilizability and detectability of the system. To check that the 8 transfer functions form a stable double coprime factorization notice first that they are all stable and that  $M_d$  e  $M_s$  are biproper systems. Finally one can use matrix calculus to verify the theorem. Alternatively, suitable state variables can be introduced. For example, from

$$\dot{x} = Ax + Bu = (A + BF)x + Bv$$

$$y = Cx + Du = (C + DF)x + Dv$$

$$u = Fx + v \quad \text{control law}$$

one obtains  $y = P(s)u = N_d(s)v$ ,  $u = M_d(s)v$ , so that  $P(s)M_d(s) = N_d(s)$ . Similarly,

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$\begin{cases} \dot{\theta} = A\theta + Bu + L\eta \\ \eta = C\theta + Du - y \end{cases} \quad \text{observer}$$

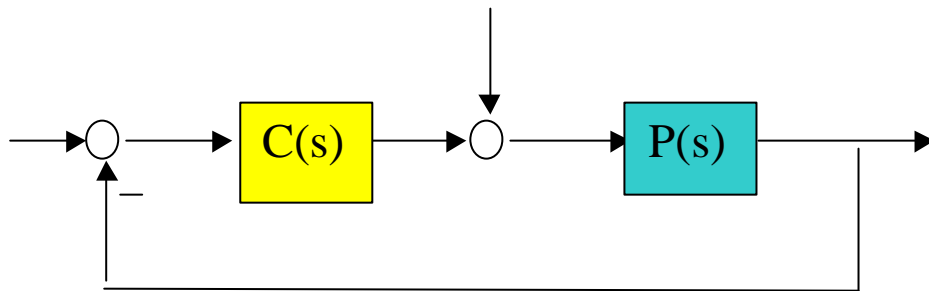
implies  $\eta = N_s(s)u - M_s(s)y = N_s(s)u - M_s(s)P(s)y = 0$  (stable autonomous dynamic) so that  $N_s(s) = M_s(s)P(s)$ . Analogously one can proceed for the rest of the proof.

# Parametrization of stabilizing controllers

## 3° case: MIMO systems and generic P(s)

$$(i) \quad P(s) = N_d(s)M_d(s)^{-1} = M_s(s)^{-1}N_s(s)$$

$$(ii) \quad \begin{bmatrix} Y_d(s) & X_d(s) \\ -N_s(s) & M_s(s) \end{bmatrix} \begin{bmatrix} M_d(s) & -X_s(s) \\ N_d(s) & Y_s(s) \end{bmatrix} = I$$



### THEOREM 5

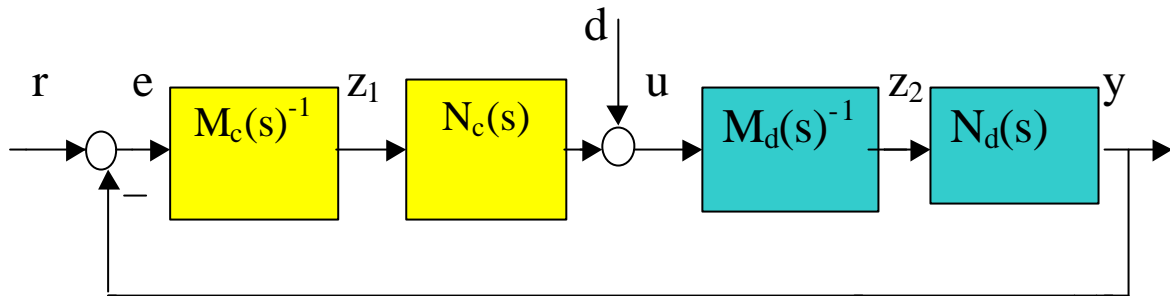
The family of all proper transfer matrices  $C(s)$  such that the closed-loop system is well-posed and asymptotically stable is:

$$\begin{aligned} C(s) &= (X_s(s) + M_d(s)Q_d(s))(Y_s(s) - N_d(s)Q_d(s))^{-1} \\ &= (Y_d(s) - Q_s(s)N_s(s))^{-1}(X_d(s) + Q_s(s)M_s(s)) \end{aligned}$$

where  $Q_d(s)$  [ $Q_s(s)$ ] is stable and such that  $N_d(\infty)Q_d(\infty) \neq Y_s(\infty)$  [ $Q_s(\infty)N_s(\infty) \neq Y_d(\infty)$ ].

## Comments

The proof of Theorem 5 follows the same lines as that of Theorem 3. However, the generalization of Lemma 1 to MIMO is required.



### LEMMA 2

Let  $P(s) = N_d(s)M_d(s)^{-1}$  and  $C(s) = N_c(s)M_c(s)^{-1}$  stable right coprime factorizations. Then the closed-loop system is asymptotically stable if and only if the transfer matrix  $K(s)$  from  $[r' \ d']'$  to  $[z_1' \ z_2']'$  is stable.

$$K(s) = \begin{bmatrix} M_c(s) & N_d(s) \\ -N_c(s) & M_d(s) \end{bmatrix}^{-1}$$

## Proof of Lemma 2

It is completely similar to the proof of Lemma 1.

## Proof of Theorem 5

Assume that  $Q(s)$  is stable and define

$$C(s) = (X_s(s) + M_d(s)Q(s))(Y_s(s) - N_d(s)Q(s))^{-1}$$

It results that

$$I = N_s(s)X_s(s) + M_s(s)Y_s(s)$$

$$= N_s(s)(X_s(s) + M_d(s)Q(s)) + M_s(s)(Y_s(s) - N_d(s)Q(s))$$

so that the functions  $X_s(s) + M_d(s)Q(s)$  e  $Y_s(s) - N_d(s)Q(s)$  defining  $C(s)$  are right coprime (besides being both stable). The four transfer matrices characterizing the closed-loop are:

$$(1+PC)^{-1} = (Y_s - N_d Q)M_s$$

$$-(1+PC)^{-1}P = (Y_s - N_d Q)N_s$$

$$(1+CP)^{-1}C = C(I+PC)^{-1} = (X_s + M_d Q)M_s$$

$$(1+CP)^{-1} = I - C(I+PC)^{-1}C = I - (X_s + M_d Q)N_s$$

They are all stable so that the closed-loop system is asymptotically stable.

Vice-versa, assume that  $C(s) = N_c(s)M_c(s)^{-1}$  (right coprime factorization) is stabilizing. Notice that the matrices

$$\begin{bmatrix} Y_d(s) & X_d(s) \\ -N_s(s) & M_s(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M_d(s) & N_c(s) \\ N_d(s) & M_c(s) \end{bmatrix} \approx K$$

have stable inverse. Hence,

$$\begin{bmatrix} Y_d(s) & X_d(s) \\ -N_s(s) & M_s(s) \end{bmatrix} \begin{bmatrix} M_d(s) & N_c(s) \\ N_d(s) & M_c(s) \end{bmatrix} = \begin{bmatrix} I & Y_d(s)N_c(s)X_d(s)N_c(s) \\ 0 & N_s(s)N_c(s) + M_s(s)M_c(s) \end{bmatrix}$$

(\*)

has stable inverse as well. Then,

$$Q(s) = (Y_d(s)N_c(s) - X_d(s)M_c(s))(M_s(s)M_c(s) + N_s(s)N_c(s))^{-1}$$

is well-posed and stable. Premultiplying equation (\*) by

$$[M_d(s) \quad -X_s(s); N_d(s) \quad Y_s(s)]$$

it follows

$$N_c(s)M_c(s)^{-1} = (X_s(s) + M_d(s)Q_d(s))(Y_s(s) - N_d(s)Q_d(s))^{-1}.$$

# Observation

Taking  $Q(s)=0$  we have the so-called *central controller*

$$C_0(s) = X_s(s)Y_s(s)^{-1}$$

which coincides with the controller designed with the *pole assignment* technique.

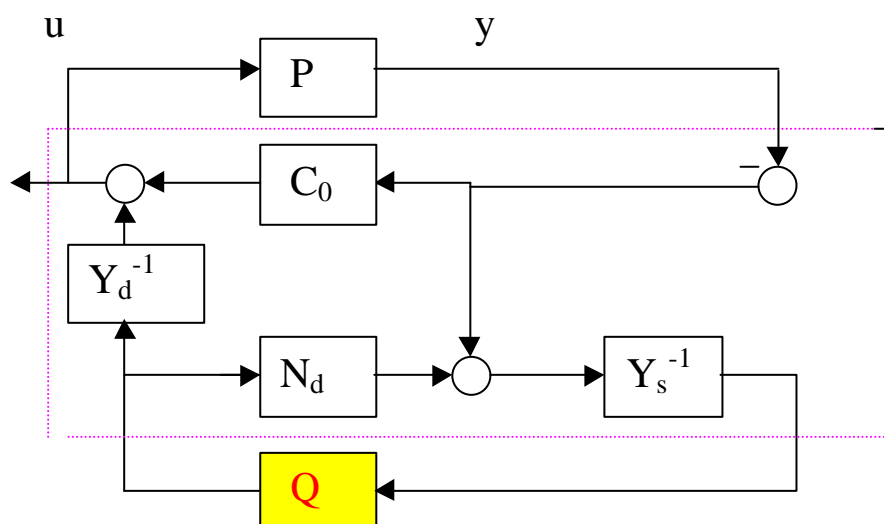
Taking  $r=d=0$  one has:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu + L(C\mathbf{x} + Du - y)$$

$$u = F\mathbf{x}$$

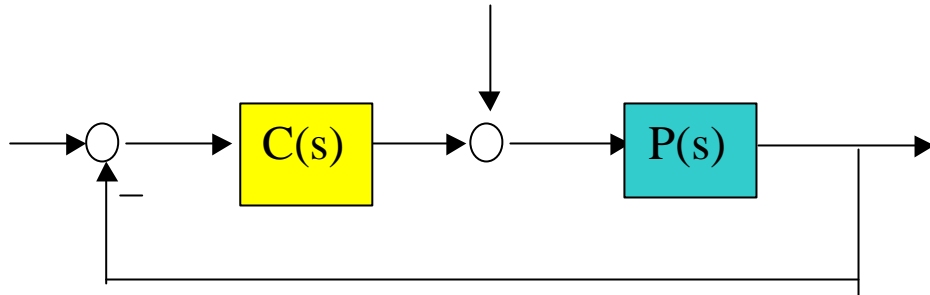
## LEMMA 3

$$C(s) = C_0(s) + Y_d(s)^{-1}Q(s)[(I - Y_s(s)^{-1}N_d(s)Q(s))]^{-1}Y_s(s)^{-1}$$





# Strong Stabilization



The problem is that of finding, if possible, a stable controller which stabilizes the closed-loop system.

## Example

$$P(s) = \frac{s-1}{(s-2)(s+2)}$$

is not stabilizable with a stable controller. Why?

$$P(s) = \frac{s-2}{(s-1)(s+2)}$$

is stabilizable with a stable controller. Why?

*Stabilization of many plants*

*Two-step stabilization*

# Interpolation

Let consider a SISO control system with  $P(s)=N(s)/M(s)$  e  $C(s)=N_c(s)/M_c(s)$  (stable coprime stabilization). Then the closed-loop system is asymptotically stable if and only if  $U(s)=N(s)N_c(s)+M(s)M_c(s)$  is an **unity** (stable with stable inverse).

Since we want that  $C(s)$  be stable, we can choice  $N_c(s)=C(s)$  and  $D_c(s)=1$ . Then,

$$C(s) = \frac{U(s) - M(s)}{N(s)}$$

Of course, we must require that  $C(s)$  be stable. This fact depends on the role of the right zeros of  $P(s)$ . Indeed, if  $N(b)=0$ , with  $\text{Re}(b) \geq 0$  ( $b$  can be infinity), then the interpolation condition must hold true:

$$U(b) = M(b)$$

Consider the first example and take  $M(s)=(s-2)(s+2)/(s+1)^2$ ,  $N(s)=(s-1)/(s+1)^2$ . Then, it must be:  $U(1)=-0.75$ ,  $U(\infty)=1$ . Obviously this is impossible.

Consider the second example and take  $M(s)=(s-1)/(s+2)$  e  $N(s)=(s-2)/(s+2)^2$ . It must be  $U(1)=0.25$ ,  $U(\infty)=1$ , which is indeed possible.

# Parity interlacing property (PIP)

## THEOREM 6

- $P(s)$  è strongly stabilizable if and only if the number of poles of  $P(s)$  between any pair of real right zeros of  $P(s)$  (including infinity) is an even number.
- We have seen that the PIP is equivalent to the existence of the existence of a unity which interpolates the right zeros of  $P(s)$ . Hence, if the PIP holds, the problem boils down to the computation of this interpolant.
- In the MIMO case, Theorem 6 holds unchanged. However it is worth pointing out that the poles must be counted accordingly with their Mc Millan degree and the zeros to be considered are the so-called blocking zeros.

## REFERENECEES

- J.G.Truxal, Control Systems Synthesis, Mc-Graw Hill, NY, 1955.
- J.R. Ragazzini, G.E. Franklin, Sampled-data Control Systems, Mc-Graw Hill, NY, 1958.
- M. Vidyasagar, Control System Synthesis: a Factorization Approach, MIT Press, Cambridge, MA, 1985.
- J.C. Doyle, B.A. Franklin, A.R. Tannenbaum, Feedback Control Theory, Mc Millan, NY, 1992.
- D.C. Youla, J.J. Bongiorno, N.N. Lu, Single-loop feedback stabilization of linear multivariable dynamical plants”, Automatica, 10, p. 159-175, 1974.
- G. Zames, “Feedback and optimal sensitivity in model reference transformation, multiplicative seminorms, approximate inverses, IEEE Trans. AC-26, p. 301-320, 1981.
- V. Kucera, Discrete Linear Control: The Polynomial Equation Approach, Wiley, NY, 1979.

# $H_2$ CONTROL

## SUMMARY

- Definition and characterization of the  $H_2$  norm
- State-feedback control (LQ)
- Parametrization, mixed problem, duality
- Disturbance rejection, Estimation (Kalman filter), Partial Information
- Conclusions and generalizations

# The $L_2$ space

$L_2$  space: The set of (rational) functions  $G(s)$  such that

$$\frac{1}{2p} \int_{-\infty}^{\infty} \text{trace} [G(-j\omega)' G(j\omega)] d\omega < \infty$$

(strictly proper – no poles on the imaginary axis!)

With the inner product of two functions  $G(s)$  and  $F(s)$ :

$$\langle G(s), F(s) \rangle = \frac{1}{2p} \int_{-\infty}^{\infty} \text{trace} [G(-j\omega)' F(j\omega)] d\omega$$

The space  $L_2$  is a pre-Hilbert space. Since it is also complete, it is indeed a Hilbert space. The norm, induced by the inner product, of a function  $G(s)$  is

$$\|G(s)\|_2 = \left[ \frac{1}{2p} \int_{-\infty}^{\infty} \text{trace} [G(-j\omega)' G(j\omega)] d\omega \right]^{1/2}$$

## The subspaces $H_2$ and $H_2^\perp$

The subspace  $H_2$  is constituted by the functions of  $L_2$  which are analytic in the right half plane. (strictly proper – stable !)

The subspace  $H_2^\perp$  is constituted by the functions of  $L_2$  which are analytic in the left half plane. (strictly proper – unstable !)

Note: A rational function in  $L_2$  is a strictly proper function without poles on the imaginary axis. A rational function in  $H_2$  is a strictly proper function without poles in the closed right half plane. A rational function in  $H_2^\perp$  is a strictly proper function without poles in the closed left half plane. The functions in  $H_2$  are related to the square integrable functions of the real variable  $t$  in  $(0, \infty]$ . The functions in  $H_2^\perp$  are related to the square integrable functions of the real variable  $t$  in  $(-\infty, 0]$ .

A function in  $L_2$  can be written in a unique way as the sum of a function in  $H_2$  and a function in  $H_2^\perp$ :  $G(s) = G_1(s) + G_2(s)$ . Of course,  $G_1(s)$  and  $G_2(s)$  are orthogonal, i.e.  $\langle G_1(s), G_2(s) \rangle = 0$ , so that

$$\|G(s)\|_2^2 = \|G_1(s)\|_2^2 + \|G_2(s)\|_2^2$$

# System theoretic interpretation of the $H_2$ norm

Consider the system

$$\dot{x}(t) = Ax(t) + Bw(t)$$

$$z(t) = Cx(t)$$

$$x(0) = 0$$

And let  $G(s)$  be its transfer function. Also consider the quantity to be evaluated:

$$J_1 = \sum_{i=1}^m \int_0^{\infty} z^{(i)'}(t) z^{(i)}(t) dt$$

where  $z^{(i)}$  represents the output of the system forced by an impulse input at the  $i$ -th component of the input.

$$J_1 = \sum_{i=1}^m \int_0^{\infty} z^{(i)'}(t) z^{(i)}(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=1}^m Z^{(i)}(-j\omega)' Z^{(i)}(j\omega) d\omega =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=1}^m e_i' G(-j\omega)' G(j\omega) e_i d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}[G(-j\omega)' G(j\omega)] d\omega = \|G(s)\|_2^2$$



# Computation of the norm Lyapunov equations

“Control-type Lyapunov equation”

$$A' P_o + P_o A + C' C = 0$$

“Filter-type Lyapunov equation”

$$P_r A' + P_r A + B B' = 0$$

$$\begin{aligned} \|G(s)\|_2^2 &= J_1 = \sum_{i=1}^m \int_0^{\infty} z^{(i)'}(t) z^{(i)}(t) dt = \int_0^{\infty} \sum_{i=1}^m \text{trace} \left[ e_i' B' e^{A't} C' C e^{At} B e_i \right] dt = \\ &= \int_0^{\infty} \text{trace} \left[ e^{A't} C' C e^{At} \sum_{i=1}^m B e_i e_i' B' \right] dt \\ &= \text{trace} \left[ B' \int_0^{\infty} e^{A't} C' C e^{At} dt B \right] = \text{trace} [B' P_o B] \\ &= \text{trace} \left[ C \int_0^{\infty} e^{At} B B' e^{A't} dt C' \right] = \text{trace} [C P_r C'] \end{aligned}$$

## Other interpretations

Assume now that  $w$  is a white noise with identity intensity and consider the quantity:

$$J_2 = \lim_{t \rightarrow \infty} E(z'(t)z(t))$$

It follows:

$$\begin{aligned} J_2 &= \lim_{t \rightarrow \infty} \text{trace} \left[ E \left[ \int_0^t C e^{A(t-t)} B w(t) dt \int_0^t w(s)' B' e^{A'(t-s)} C' ds \right] \right] E = \\ &= \lim_{t \rightarrow \infty} \text{trace} \left[ \int_0^t \int_0^t C e^{A(t-t)} B E[w(t)w(s)'] B' e^{A'(t-s)} C' dt ds \right] = \\ &= \lim_{t \rightarrow \infty} \text{trace} \left[ \int_0^t C e^{A(t-t)} B B' e^{A'(t-t)} C' dt \right] = \|G(s)\|_2^2 \end{aligned}$$

Finally, consider the quantity

$$J_3 = \lim_{T \rightarrow \infty} \frac{1}{T} E \int_0^T z'(t)z(t) dt$$

and let again  $w(\cdot)$  be a white noise with identity intensity. It follows:

$$E(z'(t)z(t)) = \text{trace}[CP(t)C'], \quad P(t) = \int_0^t e^{At} B B' e^{A't} dt$$

Notice that  $J_3 = (\|G(s)\|_2)^{1/2}$  since

$$\dot{P}(t) = AP(t) + P(t)A' + BB', \quad AY + YA' + BB' = 0, \quad Y = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T P(t) dt$$

# Optimal Linear Quadratic Control (LQ) versus Full-Information H<sub>2</sub> control

$$\dot{x} = Ax + B\bar{u}, \quad x(0) = x_0, \quad J = \int_0^{\infty} (x'Qx + 2\bar{u}'Sx + \bar{u}'R\bar{u})dt$$

Assume that

$$R > 0, \quad H = \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$$

And notice that these conditions are equivalent to

$$R > 0, \quad W = Q - SR^{-1}S' \geq 0$$

Find (if any)  $u(\cdot)$  minimizing  $J$

---

The above matrix result is the well known Schur Lemma. The proof is as follows. If  $H \geq 0$ , then

$$Q - SR^{-1}S' = \begin{bmatrix} I & -SR^{-1} \end{bmatrix} H \begin{bmatrix} I \\ -R^{-1}S' \end{bmatrix} \geq 0$$

Vice-versa, assume that  $H \geq 0$  and  $R > 0$ , and suppose by contradiction that  $W$  is not positive semidefinite, then  $Wx = vx$ , for some  $v < 0$  and  $x \neq 0$ . Then  $y'Hy = vx'x < 0$ , with  $y = \begin{bmatrix} x' \\ -x'SR^{-1} \end{bmatrix}$  is a contradiction.

# Optimal Linear Quadratic Control (LQ) versus Full-Information H<sub>2</sub> control

Let  $C_{11}$  a factorization of  $W=C_{11}'C_{11}$  and define

$$C_1 = \begin{bmatrix} C_{11} \\ R^{-1/2}S' \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$u = R^{1/2}\bar{u}, \quad z = C_1x + D_{12}u$$

Then, it is easy to verify that

$$J = \int_0^{\infty} (x'Qx + 2\bar{u}'Sx + \bar{u}'R\bar{u})dt = \int_0^{\infty} z'zdt = \|z\|_2^2$$

Moreover, the free motion of the state can be considered as a state motion caused by an impulsive input. Hence, with  $w(t)=\text{imp}(t)$ , let

$$\dot{x} = Ax + B_2u + B_1w$$

$$z = C_1x + D_{12}u$$

$$x(0) = 0$$

The (LQ) problem with stability is that of finding a controller

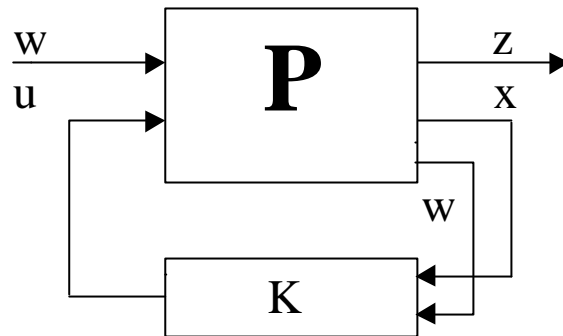
$$\dot{\mathbf{x}} = F\mathbf{x} + G_1x + G_{21}w$$

$$u = H\mathbf{x} + E_1x + E_2w$$

fed by  $x$  and  $w$  and yielding  $u$  that minimizes the H<sub>2</sub> norm of the transfer function from  $w$  to  $z$ .

- LQS problem

## Full information



$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{12} u$$

$$y = \begin{bmatrix} x \\ w \end{bmatrix}$$

Problem: Find the minimum value of  $\|T_{zw}\|_2$  attainable by an admissible controller. Find an admissible controller minimizing  $\|T_{zw}\|_2$ . Find a set of all controllers generating all  $\|T_{zw}\|_2 < \gamma$ .

### Assumptions

$(A, B_2)$  stabilizable

$D_{12}' D_{12} > 0$

$(A_c, C_{1c})$  detectable

$$A_c = A - B_2 (D_{12}' D_{12})^{-1} D_{12}' C_1$$

$$C_{1c} = (I - D_{12} (D_{12}' D_{12})^{-1} D_{12}') C_1$$

stable invariant zeros of  $(A, B_2, C_1, D_{12})$

# Solution of the Full Information Problem

## Theorem 1

There exists an admissible controller FI minimizing  $\|T_{zw}\|_2$ . It is given by

$$F_2 = -(D_{12}' D_{12})^{-1} (B_2' P + D_{12}' C_1)$$

where P is the positive semidefinite and stabilizing solution of the Riccati equation

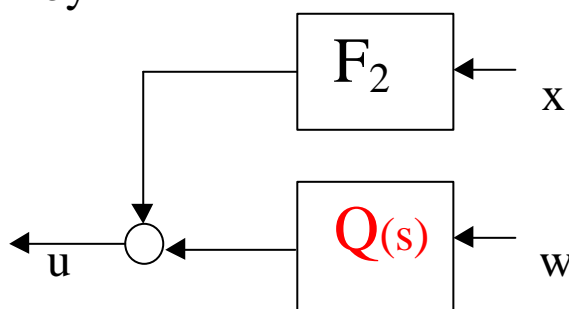
$$A'P + PA - (PB_2 + C_1' D_{12})(D_{12}' D_{12})^{-1} (B_2' P + D_{12}' C_1) + C_1' C_1 = 0$$

$$A_{cc} = A - B_2 (D_{12}' D_{12})^{-1} (B_2' P + D_{12}' C_1) = A + B_2 F_2 \quad \text{stable}$$

The minimum norm is:

$$\|T_{zw}\|_2^2 = \mathbf{a}^2, \quad \mathbf{a} = \|P_c B_1\|_2 = \sqrt{\text{trac}(B_1' P B_1)}, \quad P_c(s) = (C_1 + D_{12}' F_2)(sI - A - B_2 F_2)^{-1}$$

The set is given by

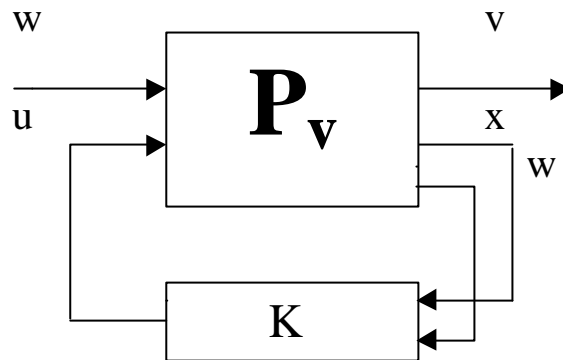


where  $Q(s)$  is a stable strictly proper system, satisfying

$$\|Q(s)\|_2^2 < \mathbf{g}^2 - \mathbf{a}^2$$

## Proof of Theorem 1

The assumptions guarantee the existence of the stabilizing solution to the Riccati equation P. Let  $v=u-F_2x$  so that  $u=F_2x+v$ , where  $v$  is a new input. Hence,  $z(s)=P_c(s)B_1w(s)+U(s)v$ , where  $U(s)=P_c(s)B_2+D_{12}$ . It follows that  $T_{zw}(s)=P_c(s)B_1+U(s)T_{vw}(s)$ . The problem is recast as to find a controller minimizing the norm from  $w$  to  $z$  of the following system



where  $P_v$  is given by:

$$\dot{x} = Ax + B_1w + B_2u$$

$$v = -F_2x + u$$

$$y = \begin{bmatrix} x \\ w \end{bmatrix}$$

Notice that  $T_{zw}(s)$  is strictly proper iff  $T_{vw}(s)$  is such. Exploiting the Riccati equality it is simple to verify that  $U(s)\tilde{U}(s)=D_{12}'D_{12}$  and that  $U(s)\tilde{P}_c(s)$  is antistable. Hence,  $\|T_{zw}(s)\|_2^2 = \|P_c(s)B_1\|_2^2 + \|T_{vw}(s)\|_2^2$ . Hence the optimal control is  $v=0$ , i.e.  $u=F_2x$ .

Finally, take a controller  $K(s)$  such that  $\|T_{zw}(s)\|_2^2 < \gamma^2$ . From this controller and the system it is possible to form the transfer function  $Q(s)=T_{vw}(s)$ . Of course, it is  $\|Q(s)\|_2^2 < \gamma^2 - \alpha^2$ . It is enough to show that the controller yielding  $u(s)=F_2x(s)+v(s)=F_2x(s)+Q(s)w(s)$  generates the same transfer function  $T_{zw}(s)$ . This computation is left to the reader.

# Disturbance Feedforward

$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{12} u$$

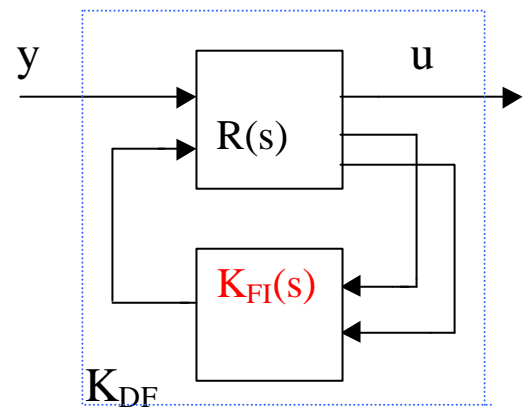
$$y = C_2 x + w$$

Same assumptions as FI+stability of  $A-B_1C_2$

$C_2=0 \rightarrow$  direct compensation

$C_2 \neq 0 \rightarrow$  indirect compensation

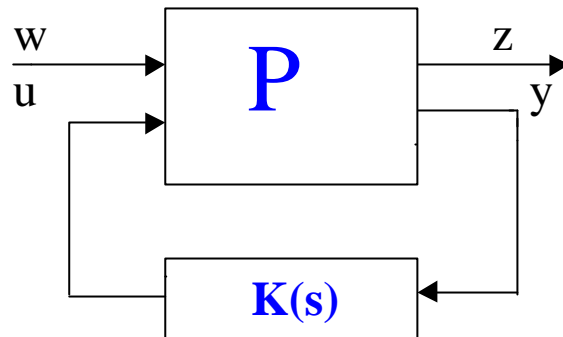
$$R(s) = \left[ \begin{array}{c|cc} A-B_1C_2 & B_1 & B_2 \\ \hline 0 & 0 & I \\ I & 0 & 0 \\ -C_2 & I & 0 \end{array} \right]$$



The proof of the DF theorem consist in verifying that the transfer function from  $w$  to  $z$  achieved by the FI regulator for the system  $A, B_1, B_2, C_1, D_{12}$  is the same as the transfer function from  $w$  to  $z$  obtained by using the regulator  $K_{DF}$  shown in the figure.



# Output Estimation

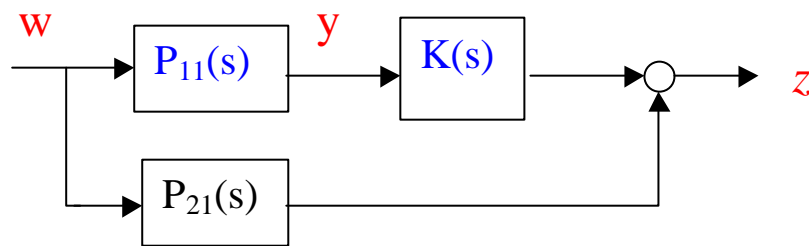


$$\dot{x} = Ax + B_1 w + (B_2 u)$$

$$u = -\hat{z}$$

$$z = C_1 x + u$$

$$y = C_2 x + D_{21} w$$



## Assumptions

$(A, C_2)$  detectable

$$D_{21} D_{21}' > 0$$

$(A_f, B_f)$  stabilizable

$(A - B_2 C_1)$  stable

$$A_f = A - B_1 D_{21}' (D_{21} D_{21}')^{-1} C_2$$

$$B_{1f} = B_1 (I - D_{21}' (D_{21} D_{21}')^{-1} D_{21})$$

# Solution of the Output Estimation problem

## Theorem 2

There exists an admissible controller (filter) minimizing  $\|T_{zw}\|_2$ . It is given by

$$\dot{\mathbf{x}} = A\mathbf{x} + B_2 u + L_2 (C_2 \mathbf{x} - y)$$

$$u = -C_1 \mathbf{x}$$

where

$$L_2 = -(PC_2 + B_2 D'_{21})(D_{21} D_{21})^{-1}$$

and P is the positive semidefinite and stabilizing solution of the Riccati equation

$$AP + PA' - (PC_2' + B_1 D_{21}') (D_{21} D_{21}')^{-1} (C_2 P + D_{21} B_1') + B_1 B_1' = 0$$

$$A_{ff} = A - (PC_2' + B_1 D_{21}') (D_{21} D_{21}')^{-1} = A + L_2 C_2 \quad \text{stable}$$

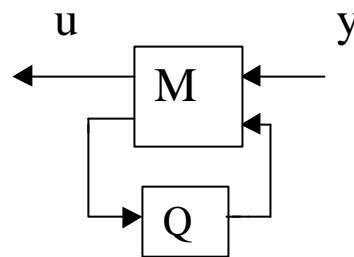
## Solution of the Output Estimation problem (cont.)

The optimal norm is

$$\|T_{zw}\|_2^2 = \|C_1 P_f(s)\|_2^2 = \sqrt{\text{trac}(C_1 P C_1')}, \quad P_f(s) = (sI - A - L_2 C_2)^{-1} (B_1 + L_2 D_{21})$$

The set of all controllers generating all  $\|T_{zw}\|_2 < \gamma$  is given by

$$M(s) = \left[ \begin{array}{c|cc} A_{ff} - B_2 C_1 & L_\infty & -B_2 \\ \hline C_1 & 0 & I \\ C_2 & I & 0 \end{array} \right]$$



Where  $Q(s)$  is proper, stable and such that  $\|Q(s)\|_2^2 < \gamma^2 - \|C_1 P_f(s)\|_2^2$

## Proof of Theorem 2

It is enough to observe that the “transpose system”

$$\dot{\mathbf{l}} = \mathbf{A}'\mathbf{l} + \mathbf{C}'_1\mathbf{V} + \mathbf{C}'_2\mathbf{n}$$

$$\mathbf{m} = \mathbf{B}'_1\mathbf{l} + \mathbf{D}'_{21}\mathbf{n}$$

$$\mathbf{J} = \mathbf{B}'_2\mathbf{l} + \mathbf{V}$$

has the same structure as the system defining the DF problem. Hence the solution is the “transpose” solution of the DF problem. It is worth noticing that the  $H_2$  solution does not depend on the particular linear combination of the state that one wants to estimate.

# Filtering

Given the system

$$\dot{x} = Ax + V_1 + B_{22}u$$

$$\bar{y} = Cx + V_2$$

where  $[z'_1 \ z'_2]'$  are zero mean white Gaussian noises with intensity

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}' & W_{22} \end{bmatrix}, \quad W_{22} > 0$$

find an estimate of the linear combination  $Sx$  of the state such as to minimize

$$J = \lim_{t \rightarrow \infty} E [(Sx(t) - u(t))'(Sx(t) - u(t))]$$

Letting

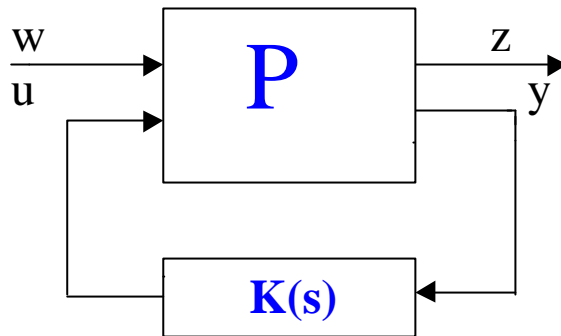
$$C_1 = -S, \quad C_2 = W_{22}^{-1}C, \quad D_{21} = [0 \ I],$$

$$B_{11}B_{11}' = W_{11} - W_{12}W_{22}^{-1}W_{12}', \quad B_1 = [B_{11} \ W_{12}W_{22}^{-1/2}]$$

$$y = W_{22}^{-1/2}\bar{y}, \quad z = C_1x + u, \quad \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} B_{11} & W_{12}W_{22}^{-1/2} \\ 0 & W_{22}^{1/2} \end{bmatrix} w$$

the problem is recast to find a controller that minimizes the  $H_2$  norm from  $w$  to  $z$ .

# The partial information problem (LQG)



$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{12} u$$

$$y = C_2 x + D_{21} w$$

Assumptions: FI + DF.

### Theorem 3

There exists an admissible controller minimizing  $\|T_{zw}\|_2$ . It is given by

$$\dot{\mathbf{x}} = A\mathbf{x} + B_2u + L_2(C_2\mathbf{x} - y)$$

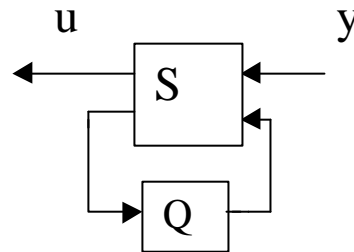
$$u = -F_2\mathbf{x}$$

The optimal norm is

$$\|T_{zw}\|_2^2 = \|P_c(s)L_2\|_2^2 + \|C_1P_f(s)\|_2^2 = \|P_c(s)B_1\|_2^2 + \|F_2P_f(s)\|_2^2 = \mathbf{g}_0^2$$

The set of all controllers generating all  $\|T_{zw}\|_2 < \gamma$  is given by

$$S(s) = \left[ \begin{array}{cc|cc} A + B_2F_2 + L_2C_2 & L_2 & -B_2 & \\ \hline -F_2 & 0 & I & \\ C_2 & I & 0 & \end{array} \right]$$



Where  $Q(s)$  is proper, stable and such that  $\|Q(s)\|_2^2 < \gamma^2 - \gamma_0^2$

#### Proof

Separation principle: The problem is reformulated as an Output Estimation problem for the estimate of  $-F_2x$ .

# Important topics to discuss

- Robustness of the LQ regulator (FI)
- Robustness of the Kalman filter (OE)
- Loss of robustness of the LQG regulator (Partial Information)
- LTR technique

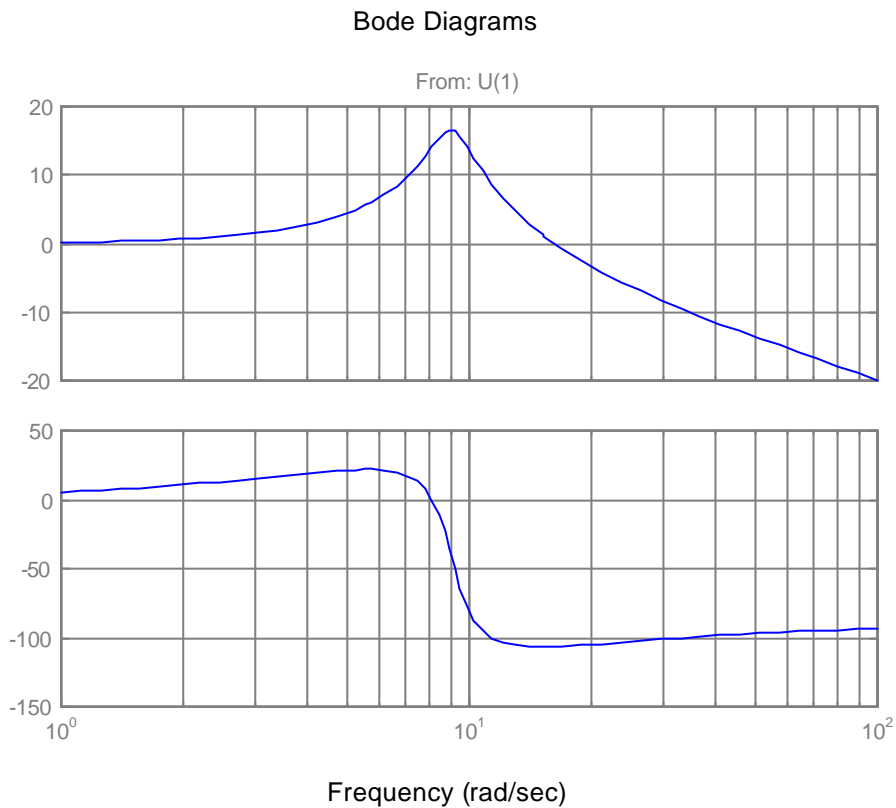


# THE $H_\infty$ CONTROL PROBLEM

## SUMMARY

- Definition and characterization of the  $H_\infty$  norm
- Description of the uncertainty: standard problem
- State-feedback control
- Parametrization, mixed problem, duality
- Disturbance rejection, Estimation, Partial Information
- Conclusions and generalizations

# What the $H_\infty$ norm is?



$G(s)$  rational stable proper

$$\|G(s)\|_\infty = \sup_{\operatorname{Re}(s) \geq 0} \|G(s)\| = \max_w \|G(jw)\|$$

$$\|\Omega\| = \bar{S}(\Omega) = \sqrt{\mathbf{I}_{\max}(\Omega * \Omega)} = \sqrt{\mathbf{I}_{\max}(\Omega \Omega^*)}$$

---

*For the frequency-domain definition we can consider a transfer function without poles on the imaginary axis. It is easy to understand that the norm of a function in  $L_\infty$  can be recast to the norm of its stable part given through the so-called inner-outer factorization. The time-domain definition, given in the next page, requires, for obvious reasons, the stability.*

# Time-domain characterization

$$G(s) = D + C(sI - A)^{-1}B$$

$$\dot{x} = Ax + Bw$$

$$z = Cx + Dw$$

A = asymptotically stable

$$\|G(s)\|_{\infty} = \sup_{w \in L_2} \frac{\|z\|_2}{\|w\|_2}$$

There does not exist a direct procedure to compute the infinity norm. However, it is easy to establish whether the norm is bounded from above by a given positive number  $\gamma$ .

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*The symbol indifferently the space of square integrable or the space of the strictly proper rational function without poles on the imaginary axis. If  $w(\cdot)$  is a white noise, the infinity norm represents the square root of the maximal intensity of the output spectrum.*

# BOUNDED REAL LEMMA

Let  $\gamma$  be a positive number and let  $A$  be asymptotically stable.

## THEOREM 1

The three following conditions are equivalent each other:

- (i)  $\|G(s)\|_{\infty} < \gamma$
- (ii)  $\|D\| < \gamma$  and there exists the positive semidefinite stabilizing solution of the Riccati equation

$$A'P + PA + (PB + C'D)(g^2I - D'D)^{-1}(B'P + DC') + C'C = 0$$

$$A + (g^2I - D'D)^{-1}(B'P + DC') \quad \text{stable}$$

- (iii)  $\|D\| < \gamma$  and there exists a positive definite solution of the Riccati inequality

$$A'P + PA + (PB + C'D)(g^2I - D'D)^{-1}(B'P + DC') + C'C < 0$$

## Comments

Notice that as  $\gamma$  tends to infinity, the Riccati equation becomes a Lyapunov equation that admits a (unique) positive semidefinite solution, thanks to the system stability. This is obvious if one notices that the infinity norm of a stable system is finite.

### Proof of Theorem 1

For simplicity, let consider the case where the system is strictly proper, i.e.  $D=0$ . The equation and the inequality are

$$A'P + PA + \frac{PBB'P}{\mathbf{g}^2} + C'C = (<)0$$

Also denote:  $G(s) \sim G(-s)'$ .

### Points (ii) $\rightarrow$ (i) and (iii) $\rightarrow$ (i).

Assume that there exists a positive semidefinite (definite) solution  $P$  of the equation (inequality). We can write:

$$(sI + A')P - P(sI - A) + \frac{PBB'P}{\mathbf{g}^2} + C'C = (<)0$$

Premultiply to the left by  $B'(sI + A')^{-1}$  e to the right by  $(sI - A)^{-1}B$ , it follows

$$\begin{aligned} G \sim(s)G(s) &= \mathbf{g}^2 I - T \sim(s)T(s) \\ T(s) &= \mathbf{g} - \mathbf{g}^{-1} B' P (sI - A)^{-1} \end{aligned}$$

so that  $\|G(s)\|_{\infty} < \gamma$ .

### Points (i) $\rightarrow$ (ii)

Assume that  $\|G(s)\|_{\infty} < \gamma$ . We now proof that the Hamiltonian matrix

$$H = \begin{bmatrix} A & \mathbf{g}^{-2} BB' \\ -C'C & -A' \end{bmatrix}$$

does not have eigenvalues on the imaginary axis. Indeed, if, by contradiction  $j\omega$  is one such eigenvalue, then

$$(j\omega - A)x - \gamma^{-2}BB'y = 0$$

$$(j\omega + A')y + C'Cx = 0$$

Hence  $Cx = -\gamma^2 C(j\omega I - A)^{-1} BB'(j\omega I + A)^{-1} C'Cx$ , so that  $G(-j\omega)'G(j\omega)Cx = 0$  and  $Cx = 0$ . Consequently,  $y = 0$  and  $x = 0$ , that is a contradiction. Then, since the Hamiltonian matrix does not have imaginary eigenvalues, it must have  $2n$  eigenvalues,  $n$  of them having negative real parts. The remaining  $n$  eigenvalues are the complex conjugate of the previous ones. This fact follows from the matrix being Hamiltonian, i.e. satisfying  $JH + H'J = 0$ , where  $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ . Let take the  $n$ -dimensional subspace generated by the (generalized) eigenvectors associated with the stable eigenvalues and let choice a matrix  $S = [X^* \ Y^*]^*$  whose range coincides with such a subspace. For a certain asymptotically stable matrix  $T$  (restriction of  $H$ ) it follows  $HS = ST$ . We now proof that  $X^*Y = Y^*X$ . Indeed let define  $V = X^*Y - Y^*X = S^*JS$  and notice that  $VT = S^*JST = S^*JHS = -S^*H'JS = -T^*S^*JS = -T^*V$ , so that  $VT + T^*V = 0$ . The stability of  $T$  yields (by the well known Lyapunov Lemma) that the unique solution is  $V = 0$ , i.e.  $X^*Y = Y^*X$ . We now proof that  $X$  is invertible. Indeed from  $HS = ST$  it follows that  $AX + \gamma^2 BB'Y = XT$  and  $-A'Y - C'CX = YT$ . Premultiplying the first equation by  $Y^*$  yields  $Y^*AX + \gamma^2 Y^*BB'Y = Y^*XT = X^*YT$ . Hence if, by contradiction,  $v \in \text{Ker}(X)$  then  $v^*Y^*BB'Yx = 0$  so that  $B'Yv = 0$ . From the first equation we have  $XTv = 0$  and  $-A'Yv = YTv$ . In conclusion, if  $v \in \text{Ker}(X)$  then  $Tv \in \text{Ker}(X)$ . By induction it follows  $T^k v \in \text{Ker}(X)$ , and again  $-A'YT^k v = YT^{k+1}v$ ,  $k$  nonnegative. Take the monic polynomial of minimum degree  $h(T)$  such that  $h(T)v = 0$  (notice that it always exists) and write  $h(T) = (\lambda I - T)m(T)$ . Since  $T$  is asymptotically stable, it results  $\text{Re}(\lambda) < 0$ . Obviously  $q = m(T)v \neq 0$ . Then  $q \in \text{Ker}(X)$ ,  $Tq = \lambda q$ ,  $-A'Yq = YTq = \lambda Yq$ . Since  $A$  is asymptotically stable and  $\text{Re}(\lambda) < 0$ , this last equation implies  $Yq = 0$ , that, together with  $Xq = 0$  implies  $q = 0$ , thanks to the  $n$ -dimensionality of the range of  $S$ . This is a contradiction. So we have proven that  $X$  is invertible. Hence, defining  $P = YX^{-1}$  and noticing that  $P^* = P$  one has  $AX + \gamma^2 BB'Y = XT$  and  $-A'Y - C'CX = YT$  so that  $-A'Y - C'CX = YX^{-1}(AX + \gamma^2 BB'Y)$  and  $A'P + C'C + PA + \gamma^2 PBB'P = 0$ .

Besides being hermitian,  $P$  is also real (and therefore symmetric). Indeed, we can write  $[X_c^* \ Y_c^*]^* N = [X^* \ Y^*]$ , where  $N$  is a permutation matrix and  $X_c, Y_c$  are complex matrices which are the complex conjugates of  $X$  e  $Y$ , respectively. Hence, if  $P_c$  is the complex conjugate of  $P$ , one has  $P = YX^{-1} = Y_c N N^{-1} X_c^{-1} = Y_c X_c^{-1} = P_c$ .

Finally, the fact that  $P$  is positive semidefinite is again a consequence of the Lyapunov Lemma, applied to the Riccati equation.

### Points (i) $\rightarrow$ (iii)

Assume that  $\|G(s)\|_\infty < \gamma$  and define

$$\bar{G}(s) = \begin{bmatrix} C \\ \sqrt{\epsilon} I \end{bmatrix} (sI - A)^{-1} B, \quad 0 < \epsilon < \frac{\gamma^2 - \|G(s)\|_\infty^2}{\|(sI - A)^{-1} B\|_\infty^2}$$

Then,

$$\bar{G}^{\sim}(s) \bar{G}(s) = G^{\sim}(s) G(s) + \epsilon F^{\sim}(s) F(s), \quad F(s) = (sI - A)^{-1} B$$

and hence

$$\|\bar{G}(s)\|_\infty^2 \leq \|G(s)\|_\infty^2 + \epsilon \|F(s)\|_\infty^2 < \gamma^2$$

Then, from the implication (i)  $\rightarrow$  (ii) it follows that there exists the positive semidefinite and stabilizing solution of the Riccati equation

$$A'P + PA + \gamma^2 PBB'P + C'C + \epsilon I = 0, \text{ so that } P > 0 \text{ and } P \text{ solves } A'P + PA + \gamma^2 PBB'P + C'C < 0.$$

# Worst Case

The “worst case” interpretation of the  $H_\infty$  norm is given by the following result:

## THEOREM 2

Let  $A$  be stable,  $\|G(s)\|_\infty < \gamma$  and let  $x_0$  be the initial state of the system. Then,

$$\sup_{w \in L_2} \|z\|_2^2 - \mathbf{g}^2 \|w\|_2^2 = x_0' P x_0$$

where  $P$  is the solution of the BRL Riccati equation.

### Proof of Theorem 2

Consider the function  $V(x) = x' P x$  and its derivative along the trajectories of the system. Letting  $\Delta = (\gamma^2 I - D' D)^{-1}$  we have:

$$\begin{aligned} \dot{V} &= x'(A' P + P A + P B w + B' P B)x \\ &= -x' C' C x - x'(P B + C' D) \Delta (B' P + D' C)x - z' z \\ &\quad + w'(D' C + B' P)x + x'(C' D + P B)w + w' D' D x + \\ &\quad - x'(P B + C' D) \Delta (B' P + D' C)x = \\ &= -z' z + \mathbf{g}^2 w' w - (w - w_{ws})' \Delta^{-1} (w - w_{ws}) \end{aligned}$$

where  $w_{ws} = \mathbf{D}(B' P + D' C)x$  is the worst disturbance. Recalling that the system is asymptotically stable and taking the integral of both hands, the conclusion follows.

## Observation: LMI

$$P > 0$$

$$A'P + PA + (PB + C'D)(\mathbf{g}^2 I - D'D)^{-1}(B'P + DC') + C'C < 0$$



*Schur Lemma*

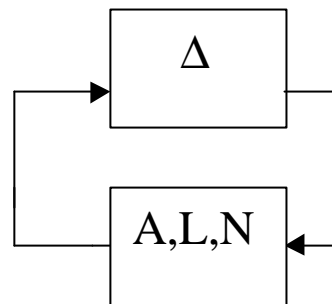
$$\begin{bmatrix} -A'P - PA - C'C & PB + C'D \\ B'P + D'C & \mathbf{g}^2 I - D'D \end{bmatrix} > 0$$



# Complex Stability Radius and quadratic stability

$$\dot{x} = (A + L\Delta N)x, \quad \|\Delta\| \leq \mathbf{a}$$

$$\begin{cases} \dot{x} &= Ax + Lw \\ z &= Nx \\ w &= \Delta x \end{cases}$$



The system is said to be quadratically stable if there exists a solution (Lyapunov function) to the following inequality, for every  $\Delta$  in the set

$$(A + L\Delta N)'P + P(A + L\Delta N) < 0$$

## **THEOREM 3**

The system is quadratically stable if and only if  $\|N(sI-A)^{-1}L\|_{\infty} < \alpha^{-1}$

### Proof of Theorem 3

First observe that the following inequality holds:

$$\begin{aligned} (A + L\Delta N)'P + P(A + L\Delta N) &= \\ A'P + PA + N'\Delta'\Delta N + PLL'P - (N'\Delta' - PL)(\Delta N - L'P) & \\ \leq A'P + PA + PLL'P + N'Na^2 = a^2(A'X + XA + \frac{XLL'X}{a^{-2}} + N'N) & \end{aligned}$$

where  $P\alpha^2 = X$ . Hence, if there exists  $X > 0$  satisfying

$$A'X + XA + \frac{XLL'X}{a^{-2}} + N'N < 0$$

then  $A + L\Delta N$  is asymptotically stable for every  $\Delta$ ,  $\|\Delta\| \leq \alpha$ , with the same Lyapunov function (quadratic stability). This happens if  $\|N(sI - A)^{-1}L\|_\infty < \alpha^{-1}$ . In conclusion, we have proven that the condition  $\|N(sI - A)^{-1}L\|_\infty < \alpha^{-1}$  implies that the system is quadratically stable. Vice-versa assume that that the system is quadratically stable. In particular the system is robustly stable, i.e. stable for each  $\Delta$  in the set  $\|\Delta\| \leq \alpha$ . Hence, for each  $\|\Delta\| < \alpha$  it results

$$(*) \quad \det[I - \Delta'G(-s)'G(s)\Delta] \neq 0, \quad \text{Re}(s) \geq 0$$

Assume by contradiction that  $\|G(s)\|_\infty \geq \alpha^{-1}$ , i.e. there exists  $b$  such that

$$\mathbf{I}_{\max} [I - \mathbf{a}G(-jb)'G(jb)\mathbf{a}] \leq 0$$

Since  $\lambda_{\max}[I - \alpha G(-\infty)'G(\infty)\alpha] = 1 > 0$ , we have that there exists  $s = j\omega$  that violates (\*), a contradiction.

# Real stability radius

$A$  = stable (eigenvalues with strictly negative real part)

$$\begin{aligned} r_r(A, B, C) &= \inf \left\{ \|\Delta\| : \Delta \in R^{m \times p} \text{ and } A + B\Delta C \text{ is unstable} \right\} \\ &= \inf_{s \in j\omega} \inf \left\{ \|\Delta\| : \Delta \in R^{m \times p} \text{ and } \det(sI - A - B\Delta C) = 0 \right\} \\ &= \inf_{s \in j\omega} \inf \left\{ \|\Delta\| : \Delta \in R^{m \times p} \text{ and } \det(I - \Delta C(sI - A)^{-1} B) = 0 \right\} \end{aligned}$$

**Linear algebra problem: compute**

$$m_r(M) = \left[ \inf \left\{ \|\Delta\| : \Delta \in R^{m \times p} \text{ and } \det(I - \Delta M) = 0 \right\} \right]^{-1}$$

Proposition

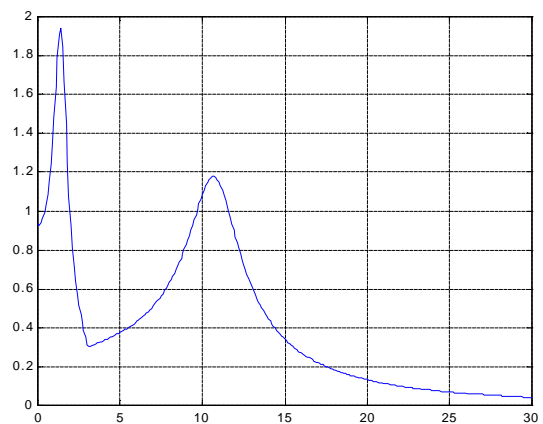
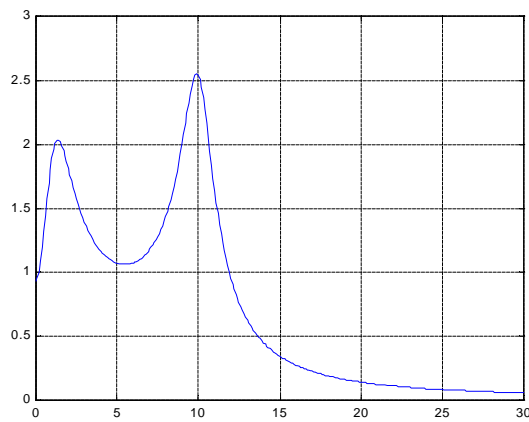
$$m_r(M) = \inf_{g \in (0,1]} \mathbf{S}_2 \left( \begin{bmatrix} \operatorname{Re} M & -g \operatorname{Im} M \\ g^{-1} \operatorname{Im} M & \operatorname{Re} M \end{bmatrix} \right)$$

**Taking  $M=G(s)=C(sI-A)^{-1}B$  it follows**

$$r_r(A, B, C)^{-1} = \sup_w \inf_{g \in (0,1]} \mathbf{S}_2 \left( \begin{bmatrix} \operatorname{Re} G(j\omega) & -g \operatorname{Im} G(j\omega) \\ g^{-1} \operatorname{Im} G(j\omega) & \operatorname{Re} G(j\omega) \end{bmatrix} \right)$$

# Example

$$A = \begin{bmatrix} 79 & 20 & -30 & -20 \\ -41 & -12 & 17 & 13 \\ 167 & 40 & -60 & -38 \\ 33.5 & 9 & -14.5 & -11 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2190 & 0.9347 \\ 0.0470 & 0.3835 \\ 0.6789 & 0.5194 \\ 0.6793 & 0.8310 \end{bmatrix}$$
$$C = \begin{bmatrix} 0.0346 & 0.5297 & 0.0077 & 0.0668 \\ 0.0535 & 0.6711 & 0.3834 & 0.4175 \end{bmatrix}$$



$$\Delta_{\text{worst}} = \begin{bmatrix} -0.4996 & 0.1214 \\ 0.1214 & 0.4996 \end{bmatrix}$$

# Entropy

Consider a stable system  $G(s)$  with state space realization  $(A,B,C,0)$ , and assume that  $\|G(s)\|_\infty < \gamma$ .

The  $\gamma$ -entropy of the system is defined as

$$I_g(G) = -\frac{g^2}{2p} \int_{-\infty}^{+\infty} \ln \det \left[ I - \frac{G(-j\omega)'G(j\omega)}{g^2} \right] d\omega$$

## Proposition

$$\|G(s)\|_2^2 \leq I_g(G) \leq \left( \frac{-\log(1-a^2)}{a^2} \right) \|G(s)\|_2^2, \quad a = \frac{\|G(s)\|_\infty}{g}$$

$$I_g(G) = \text{trace} [B'PB] = \text{trace} [CQC']$$

Where  $P$  and  $Q$  are the stabilizing solutions of the Riccati equations

$$A'P + PA + g^{-2}PBB'P + C'C = 0$$

$$AQ + QA' + g^{-2}QC'CP + BB' = 0$$

## Comments

It is easy to check that the  $\gamma$ -entropy measure is not a norm, but can be considered as a generalization of the square of the  $H_2$  norm

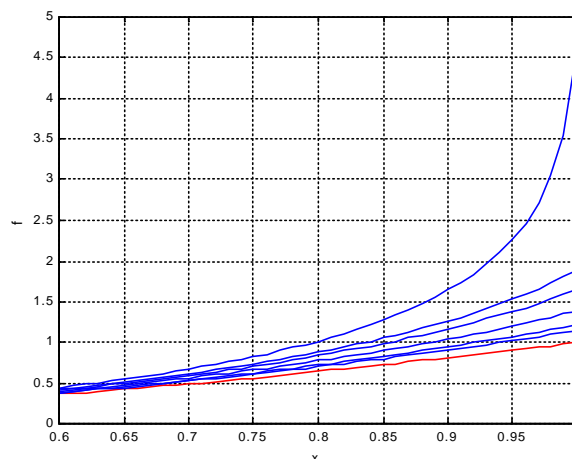
$$\|G(s)\|_2^2 = \frac{1}{2p} \int \sum_{-\infty}^{+\infty} \mathbf{s}_i^2 [G(j\omega)] d\omega$$

Indeed the  $\gamma$ -entropy can be written as

$$I_g(G) = \frac{1}{2p} \int \sum_{-\infty}^{+\infty} f(\mathbf{s}_i^2 [G(j\omega)]) d\omega$$

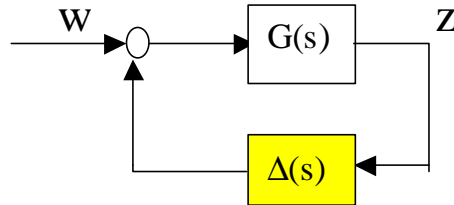
$$f(x^2) = -g^2 \ln \left( 1 - \frac{x^2}{g^2} \right)$$

Graph of  $f(x^2)$  parametrized in  $\gamma > 1$ . For large  $\gamma$  the function  $f(x^2)$  gets closer to  $x^2$  (red line).



## Comments

An interpretation for the  $\gamma$ -entropy for SISO systems is as follows. Consider the feedback configuration:



where  $w(\cdot)$  is a white noise and  $\Delta(s)$  is a random transfer function with  $\Delta(j\omega_1)$  independent on  $\Delta(j\omega_2)$  and uniformly distributed on the disk of radius  $\gamma^{-1}$  in the complex plane.

Hence the expectation value over the random feedback transfer function is

$$E_{\Delta} \left( \left\| \frac{G(s)}{1 - G(s)\Delta(s)} \right\|_2^2 \right) = I_g(G)$$

## Proof of the Proposition

First notice that  $f(x^2) \geq x^2$  so that the conclusion that  $I_g(G) \geq \|G(s)\|_2^2$  follows immediately. Now, let

$$\mathbf{b} = \frac{1}{\mathbf{a}^2} = \frac{\mathbf{g}^2}{\|G(s)\|_\infty^2}, \quad r_i = \frac{\mathbf{g}^2}{\mathbf{s}_i^2(G(j\mathbf{w}))}$$

Of course it is  $r_i \geq \beta > 1$ . Then

$$\left(1 - \frac{1}{r_i}\right)^{r_i} \geq \left(1 - \frac{1}{\mathbf{b}}\right)^{\mathbf{b}} \quad \longrightarrow \quad \ln\left(1 - \frac{1}{r_i}\right) \geq \frac{\mathbf{b}}{r_i} \ln\left(1 - \frac{1}{\mathbf{b}}\right)$$

so that

$$I_g(G) = \frac{1}{2\mathbf{p}} \int \sum_{i=1}^{+\infty} -\mathbf{g}^2 \ln\left(1 - \frac{1}{r_i}\right) d\mathbf{w} \leq -\mathbf{b} \ln\left(1 - \frac{1}{\mathbf{b}}\right) \frac{1}{2\mathbf{p}} \int \sum_{i=1}^{+\infty} \frac{\mathbf{g}^2}{r_i} d\mathbf{w}$$

which is the conclusion. In order to prove that the entropy can be computed from the stabilizing solution of the Riccati equation, recall (Theorem 1) that, since  $\|G(s)\|_\infty < \gamma$ , it is possible to write

$$\tilde{G}^*(s)G(s) = \mathbf{g}^2 I - T^*(s)T(s)$$

$$T(s) = \mathbf{g} - \mathbf{g}^{-1} B' P (sI - A)^{-1}$$

where P is the stabilizing solution of the Riccati equation (with unknown P). Then, letting  $\gamma \Omega(s) = T(s)$  it follows

$$\begin{aligned} I_g(G) &= -\frac{\mathbf{g}^2}{2\mathbf{p}} \int_{-\infty}^{+\infty} \log \det [\Omega(-j\mathbf{w})' \Omega(j\mathbf{w})] d\mathbf{w} = \lim_{z \rightarrow \infty} -\frac{\mathbf{g}^2}{2\mathbf{p}} \int_{-\infty}^{+\infty} \log \det [\Omega(-j\mathbf{w})' \Omega(j\mathbf{w})] \frac{|z^2|}{|(z - j\mathbf{w})|^2} d\mathbf{w} \\ &= \lim_{z \rightarrow \infty} -\frac{\mathbf{g}^2}{\mathbf{p}} \int_{-\infty}^{+\infty} \log |\det \Omega(j\mathbf{w})| \frac{|z^2|}{|(z - j\mathbf{w})|^2} d\mathbf{w} \\ I_g(G) &= \lim_{z \rightarrow \infty} -\mathbf{g}^2 z \log |\det \Omega(z)| = \lim_{z \rightarrow \infty} -\mathbf{g}^2 z \log |\det (I - \mathbf{g}^{-2} B' P (zI - A)^{-1} B| \\ &= \lim_{z \rightarrow \infty} -\mathbf{g}^2 z \log \left| \det \left( I - \frac{\mathbf{g}^{-2}}{z} B' P (B + M(z^{-1})) \right) \right| = \lim_{z \rightarrow \infty} -\mathbf{g}^2 z \log \left| \left( 1 - \frac{\mathbf{g}^{-2}}{z} \text{trace}[B' P (B + M(z^{-1}))] \right) \right| \\ &= \lim_{z \rightarrow \infty} -\mathbf{g}^2 z \log \left| \left( 1 - \frac{\mathbf{g}^{-2}}{z} \text{trace}[B' P B] + O(z^{-2}) \right) \right| = \lim_{z \rightarrow \infty} -\mathbf{g}^2 z \left( -\frac{\mathbf{g}^{-2}}{z} \text{trace}[B' P B] + O(z^{-2}) \right) \\ &= \text{trace}[B' P B] \end{aligned}$$

In view of the Poisson integral formula it follows that

In the expressions above we set  $M(z^{-1}) = ABz^{-1} + A^2Bz^{-2} + \dots$ . Moreover,  $O(z^{-2})$  denotes terms of powers  $z^{-2}, z^{-3}$  ect.. Finally, the formulas  $\det(I + \varepsilon V) = 1 + \varepsilon \text{trace}(V) + O(\varepsilon^2)$  and  $\log \det(I + \varepsilon V) = \varepsilon \text{trace}(V) + O(\varepsilon^2)$  have been used. The proof of the proposition with the solution of the Riccati equation with unknown Q follows the same lines and therefore is omitted.



# ACTUATOR DISTURBANCE

$$\dot{x} = Ax + B(w + u)$$

$$z = Cx + Du$$

The optimal  $H_2$  state-feedback controller is

$$u = F_2 x, \quad F_2 = -(D'D)^{-1}(B'P + D'C)$$

$$0 = A'P + PA + C'C - F_2 D' D F_2$$

## Proposition

The optimal  $H_2$  control law ensures an  $H_\infty$  norm of the closed loop system (from  $w$  to  $z$ ) lower than  $2\|D\|$ .

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Proof. Consider the equation of the Bounded Real Lemma for the closed-loop system, i.e.

$$(A + BF_2)'Q + Q(A + BF_2) + \mathbf{g}^{-2}QBB'Q + (C + DF_2)'(C + DF_2) < 0$$

Now take  $Q=P/\alpha$  and consider both equations in  $P$ . It follows:

$$(1 - \mathbf{a}^{-1})C'(I - D(D'D)^{-1}D')C + PBRB'P < 0, \quad R = (D'D)^{-1}(1 - \mathbf{a}^{-1}) + I\mathbf{a}^{-1}\mathbf{g}^{-2}$$

This condition is verified by choosing  $\alpha \leq 1$  and the minimum  $\gamma$  such that  $R \leq 0$ , i.e. the minimum  $\gamma$  such that

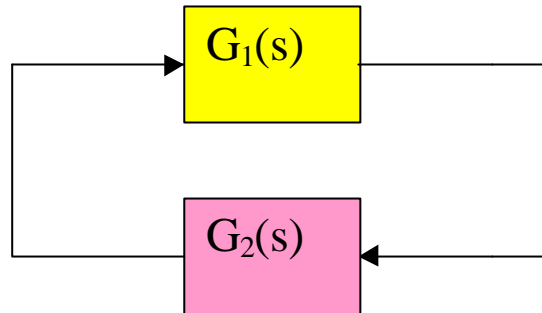
$$(\mathbf{a} - \mathbf{a}^2)I - \mathbf{g}^{-2}D'D \geq \mathbf{a} - \mathbf{a}^2 - \mathbf{g}^{-2}\bar{\mathbf{s}}^2(D) = 0$$

$\alpha - \alpha^2$

The term  $\alpha - \alpha^2$  is maximized by  $\alpha = 0.5$ , for which  $\alpha - \alpha^2 = 0.25$ , so that the minimum  $\gamma$  is

$$\mathbf{g} = 2\bar{\mathbf{s}}(D)$$

# The Small Gain Theorem



## **THEOREM 4**

Assume that  $G_1(s)$  is stable. Then:

- (i) The interconnected system is stable for each stable  $G_2(s)$  with  $\|G_2(s)\|_\infty < \alpha$  if  $\|G_1(s)\|_\infty \leq \alpha^{-1}$ .
- (ii) If  $\|G_1(s)\|_\infty > \alpha^{-1}$  then there exists a stable  $G_2(s)$  with  $\|G_2(s)\|_\infty < \alpha$  that destabilizes the interconnected system.

(iii) Proof of Theorem 4

Point (i).

If  $\|G_2(s)\|_\infty < \alpha$  and  $\|G_1(s)\|_\infty \leq \alpha^{-1}$ , then  $\det[I - G_1(s)G_2(s)] \neq 0$ , for  $\text{Re}(s) \geq 0$ . This fact, together with the stability of  $G_1(s)$  and  $G_1(s)$ , is equivalent to the stability of the closed-loop system (the simple check is left to the reader).

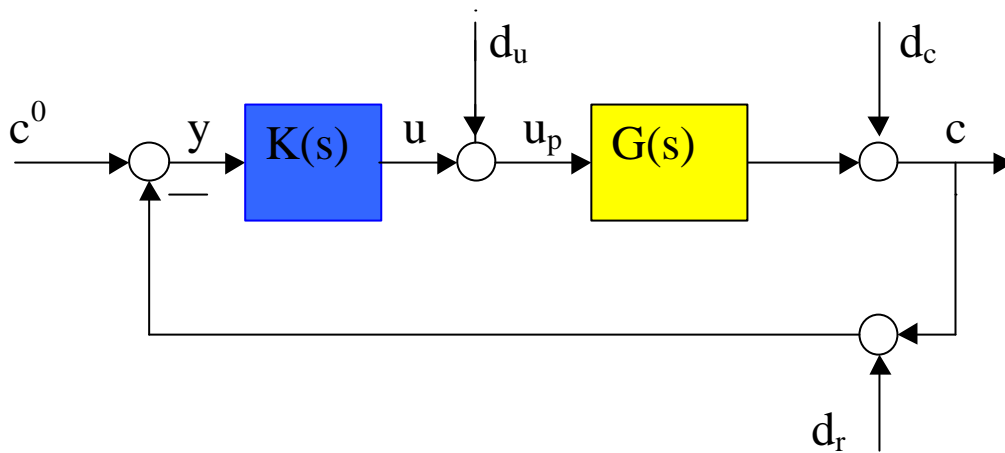
Point (ii).

For the proof of this theorem, let consider the case where the number  $m$  of column of  $G_1(s)$  is less than or equal to the number  $p$  of columns of  $G_2(s)$ . The proof in the converse case is similar. Then, assume that  $\|G_1(s)\|_\infty = \alpha^{-1}(1 + \varepsilon) = \rho^{-1}$ ,  $\varepsilon > 0$ , and write the singular value decomposition of  $G_1(j\omega)$ , i.e.  $G_1(j\omega) = U(j\omega)\Sigma(j\omega)V^{\sim}(j\omega)$  where  $\Sigma(j\omega) = [S(j\omega)' \ 0]'$  and  $S(j\omega)$  is square with dimension  $m$ . Moreover, take a stable  $G_2(s)$  such that  $G_2(j\omega) = \rho V(j\omega)[I \ 0]U^{\sim}(j\omega)$ . Notice that  $G_2^{\sim}(j\omega)G_2(j\omega) = \rho^2 < \alpha^2$  so that  $\|G_2(s)\|_\infty < \alpha$ . We have:

$$\begin{aligned} \det[I - G_1(j\omega)G_2(j\omega)] &= \det \left[ I - \mathbf{r}U(j\omega) \begin{bmatrix} S(j\omega) & 0 \\ 0 & 0 \end{bmatrix} U^{\sim}(j\omega) \right] \\ &= \det \left[ \begin{bmatrix} U(j\omega)U^{\sim}(j\omega) - \mathbf{r}U(j\omega) \begin{bmatrix} S(j\omega) & 0 \\ 0 & 0 \end{bmatrix} U^{\sim}(j\omega) \end{bmatrix} \right] \\ &= \det \left[ \begin{bmatrix} I - \mathbf{r}S(j\omega) & 0 \\ 0 & I \end{bmatrix} \right] = \det[I - \mathbf{r}S(j\omega)] \end{aligned}$$

Since  $\|G_1(s)\|_\infty = \rho^{-1}$ , it follows that there exists a frequency  $b$  such that  $\lim_{\omega \rightarrow b} \sigma(G_1(j\omega)) = \rho^{-1}$ . Hence, being  $S(j\omega)$  diagonal with the singular values of  $G_1(j\omega)$  on the diagonal, it follows that at least one entry of  $\rho S(j\omega)$  goes to zero as  $\omega$  tends to  $b$ . In conclusion  $\lim_{\omega \rightarrow b} \det[I - \rho S(j\omega)] = 0$ , so that the closed-loop system is not stable.

# The $H_\infty$ design problem



Find  $K(s)$  in such a way to guarantee:

- Stability
- Satisfactory performances

*Design in nominal conditions*

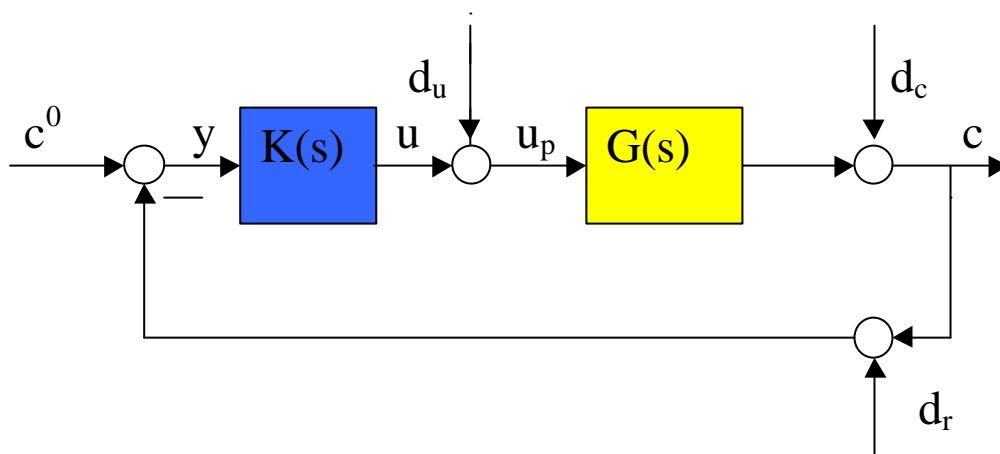
*Uncertainties description*

*Design under uncertain conditions*

# Nominal Design

$$G(s) = G_n(s)$$

The performances are expressed in terms of requirements on some transfer functions



Characteristic functions:

- Sensitivity  $S_n(s) = (I + G_n(s)K(s))^{-1}$   
 $d_c \rightarrow c$ ,  $c^0 \rightarrow y$ ,  $-d_r \rightarrow y$
- Complementary sensitivity  
 $T_n(s) = G_n(s)K(s)(I + G_n(s)K(s))^{-1}$   
 $c^0 \rightarrow c$ ,  $-d_r \rightarrow c$
- Control sensitivity  
 $V_n(s) = K(s)(I + G_n(s)K(s))^{-1}$   
 $c^0 \rightarrow u_p$ ,  $-d_r \rightarrow u_p$ ,  $-d_c \rightarrow u_p$

# Shaping Functions

In the SISO case, the requirement to have a “small” transfer function  $\mathbf{f}(s)$  can be well expressed by saying that the absolute value  $|\mathbf{f}(j\omega)|$ , for each frequency  $\omega$ , must be smaller than a given function  $\mathbf{q}(\omega)$  (generally depending on the frequency):

$$|\mathbf{f}(j\omega)| < \mathbf{q}(\omega), \quad \forall \omega$$

Analogously, in the MIMO case, one can write

$$\bar{\mathbf{s}}[\mathbf{f}(j\omega)] < \mathbf{q}(\omega), \quad \forall \omega$$

This can be done by choosing a matrix transfer function  $\mathbf{W}(s)$ , stable with stable inverse (“**shaping**” function), such that

$$\forall \omega \quad \bar{\mathbf{s}}[\mathbf{W}^{-1}(j\omega)] = \mathbf{q}(\omega) \leftarrow \|\mathbf{W}(s)\mathbf{f}(s)\|_{\infty} < 1$$

A general requirement is that the sensitivity function is small at low frequency (tracking) whereas the complementary sensitivity function is small at high frequency (feedback disturbances attenuation). **Mixed sensitivity**:

$$\left\| \begin{bmatrix} W_1(s)S_n(s) \\ W_2(s)T_n(s) \end{bmatrix} \right\|_{\infty} < 1$$

## Description of the uncertainties

The nominal transfer function  $G_n(s)$  of the process  $G(s)$  belongs to a set  $\mathbf{G}$  which can be parametrized through a transfer function  $\Delta(s)$  included in the set

$$D_a = \{ \Delta(s) \mid \Delta(s) \in H_\infty, \|\Delta(s)\|_\infty < a \}$$

For instance,

- $\mathbf{G} = \{ G(s) \mid G(s) = G_n(s) + \Delta(s) \}$
- $\mathbf{G} = \{ G(s) \mid G(s) = G_n(s)(\mathbf{I} + \Delta(s)) \}$
- $\mathbf{G} = \{ G(s) \mid G(s) = (\mathbf{I} + \Delta(s))G_n(s) \}$
- $\mathbf{G} = \{ G(s) \mid G(s) = (\mathbf{I} - \Delta(s))^{-1}G_n(s) \}$
- $\mathbf{G} = \{ G(s) \mid G(s) = (\mathbf{I} - G_n(s)\Delta(s))^{-1}G_n(s) \}$

## Examples

- $\mathbf{G} = \{G(s) \mid G(s) = G_n(s) + \Delta(s)\}$

### Uncertain unstable zeros

$$G(s) = \frac{s-2}{(s+2)(s+1)} + \frac{\mathbf{e}}{(s+2)(s+1)}$$

- $\mathbf{G} = \{G(s) \mid G(s) = G_n(s)(\mathbf{I} + \Delta(s))\}$

### Unmodelled high frequency poles or unstable zeros

$$G(s) = \frac{1}{(s+1)} \left( 1 - \frac{\mathbf{e}s}{(1+\mathbf{e}s)} \right) \quad G(s) = \frac{1}{(s+1)} \left( 1 - \frac{2}{(1+s)} \right)$$

- $\mathbf{G} = \{G(s) \mid G(s) = (\mathbf{I} - \Delta(s))^{-1} G_n(s)\}$

### Unmodelled unstable poles

$$G(s) = \left( 1 - \frac{10}{(1+s)} \right)^{-1} \frac{1}{(s+10)}$$

- $\mathbf{G} = \{G(s) \mid G(s) = (\mathbf{I} - G_n(s)\Delta(s))^{-1} G_n(s)\}$

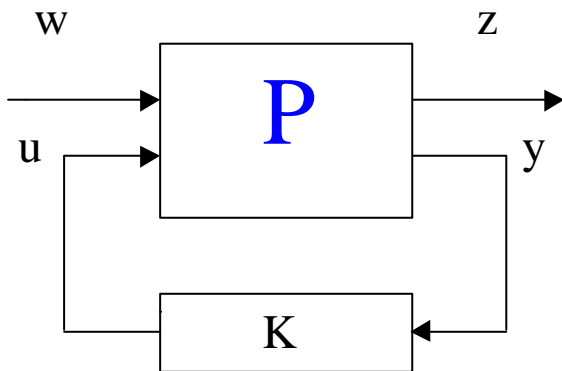
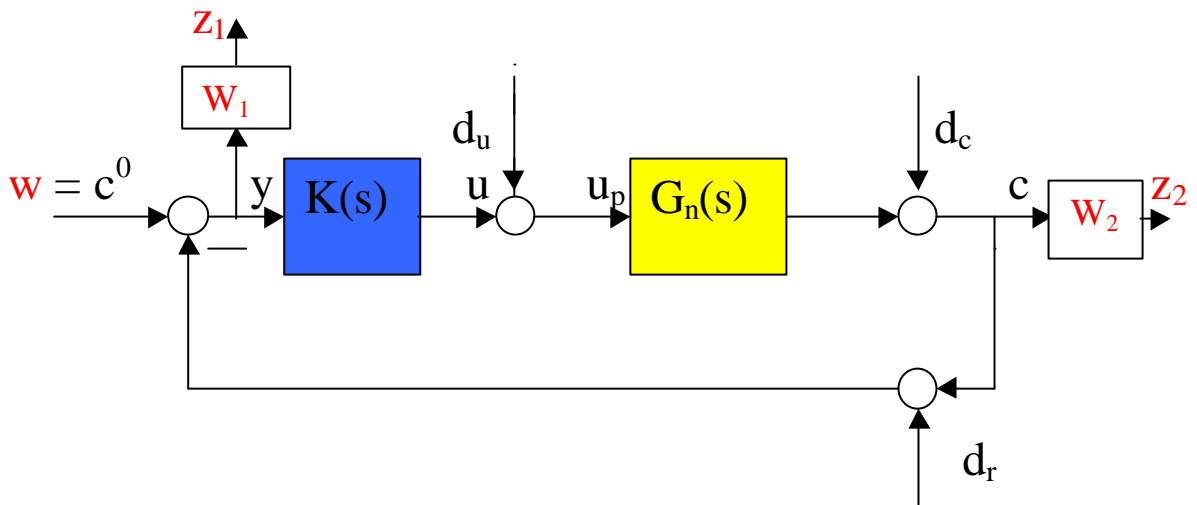
### Uncertain unstable poles

$$G(s) = \left( 1 - \frac{1}{s-1} \mathbf{e} \right)^{-1} \frac{1}{(s-1)}$$



# Design in nominal conditions

## Example of mixed performances



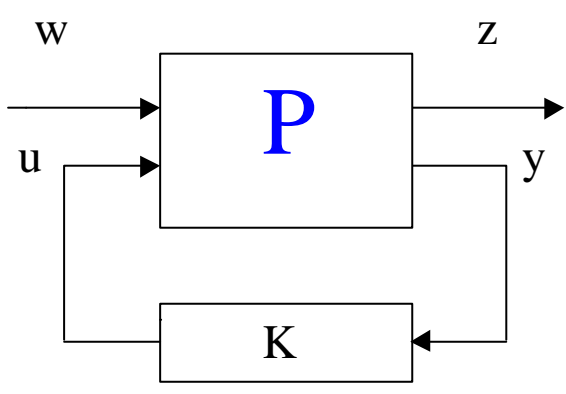
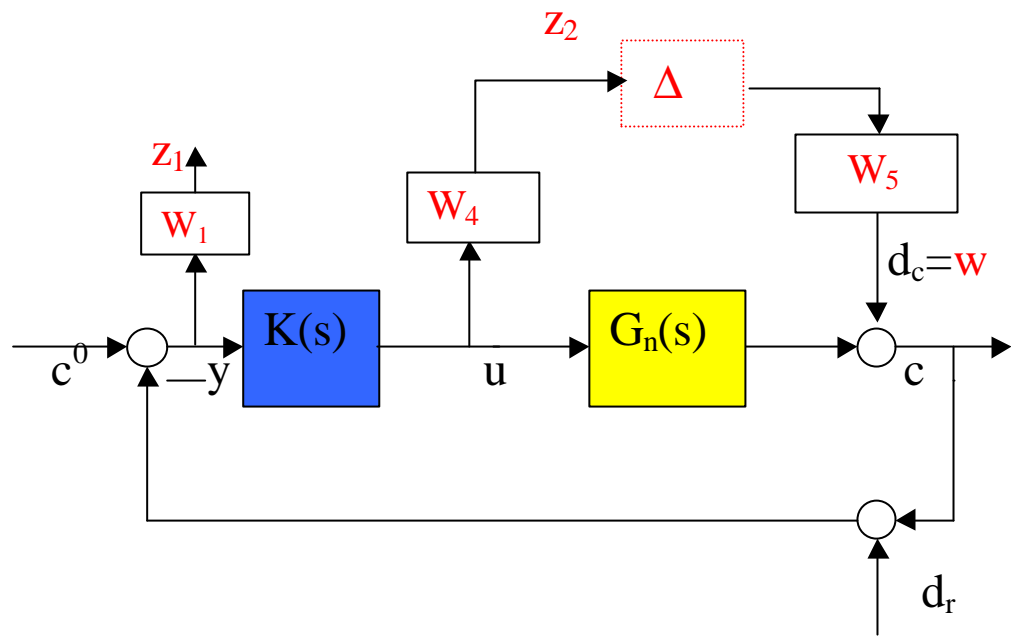
$$W_5=1$$

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, w = c^0$$

$$P = \begin{bmatrix} W_1 & -W_1 G_n \\ 0 & W_2 G_n \\ I & -G_n \end{bmatrix}$$

# Design under uncertain condition

## Robust stability and nominal performances

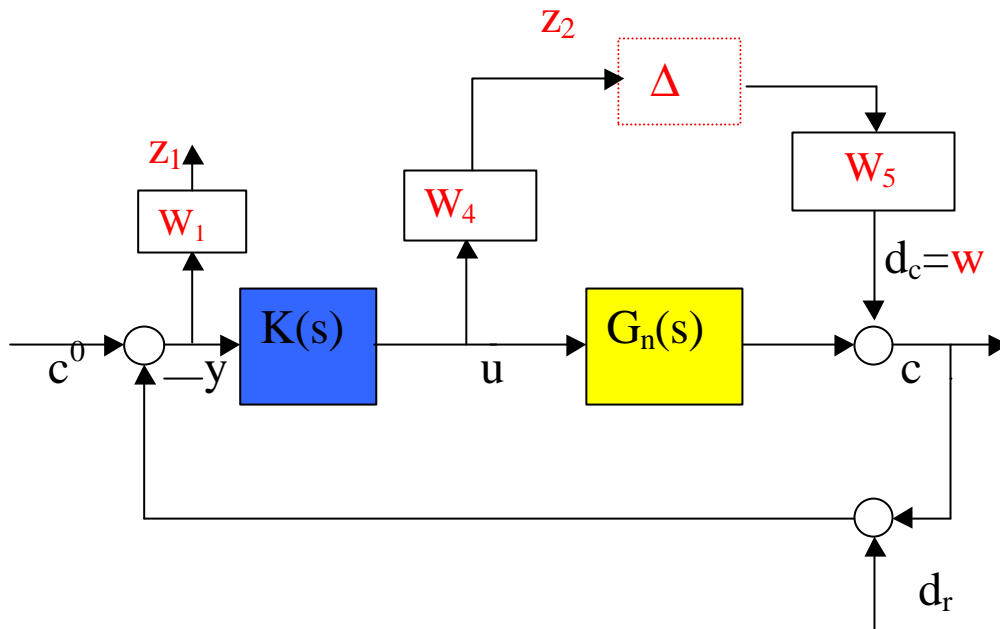


$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, w = d_c$$

$$P = \begin{bmatrix} -W_1 & -W_1 G_n \\ 0 & W_4 G_n \\ -I & -G_n \end{bmatrix}$$

# Design under uncertain condition

## Robust stability and robust performances



### 1) Robust sensitivity performance

$$\|W_1(s)S(s)\|_\infty = \left\| W_1(s)S_n(s) \left[ I - W_5(s)\Delta(s)W_4(s)V_n(s) \right]^{-1} \right\|, \quad \forall \|\Delta(s)\|_\infty < 1$$

### 2) Robust stability

$$\|W_4(s)V_n(s)W_5(s)\|_\infty < 1$$

**Result:** (take  $W_5=1$ )

1) and 2) are both achieved if the controller is such that

$$\left\| \begin{bmatrix} W_1(s)S_n(s) \\ W_4(s)V_n(s) \end{bmatrix} \right\|_\infty < 1$$

The proof of the result follows from 1) and 2) and on the following

**Lemma**

Let  $X(s)$  and  $Y(s)$  in  $L_\infty$  and  $\Psi$  a generic  $L_\infty$  function such that  $\|Y_\infty\| < 1$ . If

$$\sup_w \left\{ \|X(j\omega)\| + \|Y(j\omega)\| \right\} < 1$$

Then

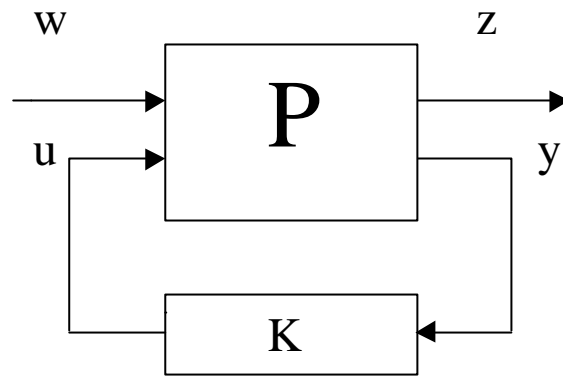
$$\begin{aligned} \sup_w \|X(j\omega)\| &< 1 \\ \sup_w \left\{ \|Y(j\omega)[I - \Psi(j\omega)X(j\omega)]^{-1}\| \right\} &< 1 \end{aligned}$$

**Proof of the Lemma**

The first point directly follows from the assumption. The second point is trivial if  $X=0$ . Hence, assume  $X \neq 0$  so that

$$\begin{aligned} \|Y(j\omega)[I - \Psi(j\omega)X(j\omega)]^{-1}\| &\leq \|Y(j\omega)\| \| [I - \Psi(j\omega)X(j\omega)]^{-1} \| \\ &= \frac{\|Y(j\omega)\|}{\mathbf{s}_{\min}(I - \Psi(j\omega)X(j\omega))} \leq \frac{\|Y(j\omega)\|}{1 - \mathbf{s}_{\max}(\Psi(j\omega)X(j\omega))} \\ &\leq \frac{\|Y(j\omega)\|}{1 - \mathbf{s}_{\max}(\Psi(j\omega))\mathbf{s}_{\max}(X(j\omega))} \leq \frac{\|Y(j\omega)\|}{1 - \mathbf{s}_{\max}(X(j\omega))} \\ &\leq \frac{\|Y(j\omega)\|}{1 - \|X(j\omega)\|} \leq 1 \end{aligned}$$

## The Standard Problem

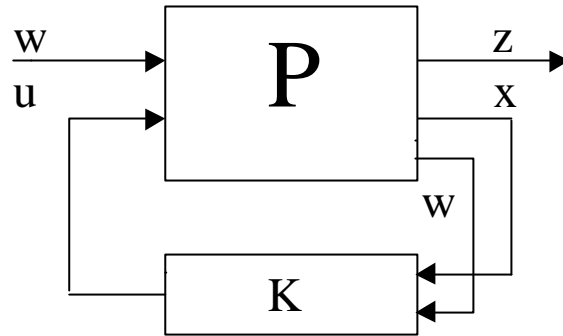


All the situations studied so far can be recast in the so-called standard problem: Find  $K(s)$  in such a way that:

- The closed-loop system is asymptotically stable
- $\|T(z, w, s)\|_{\infty} < \mathbf{g}$

***Existence of a feasible  $K(s)$  and parametrization of all controllers such that the closed-loop norm between  $w$  and  $z$  be less than  $\mathbf{g}$ .***

## Full information



$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{12} u$$

$$y = \begin{bmatrix} x \\ w \end{bmatrix}$$

### *Assumptions*

$(A, B_2)$  stabilizzabile

$D_{12}' D_{12} > 0$

$(A_c, C_{1c})$  rivelabile

$$A_c = A - B_2 (D_{12}' D_{12})^{-1} D_{12}' C_1$$

$$C_{1c} = (I - D_{12} (D_{12}' D_{12})^{-1} D_{12}') C_1$$

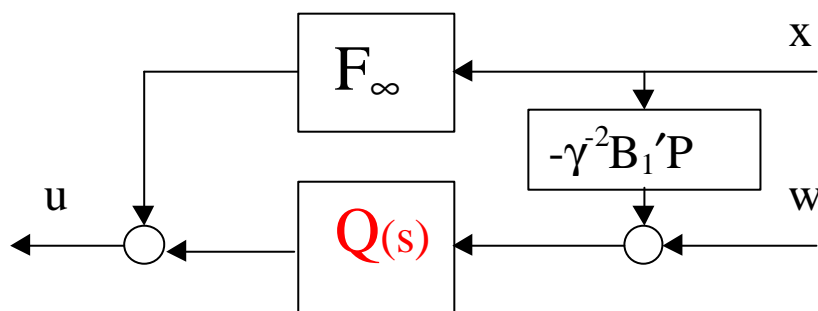
# Solution of the Full Information Problem

## THEOREM 5

There exists a controller FI, feasible and such that  $\|T(z,w,s)\|_\infty < \gamma$  if and only if there exists a positive semidefinite and stabilizing solution of the Riccati equation

$$A'P + PA - (PB_2 + C_1'D_{12})(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1) + P\frac{B_1B_1'}{g^2}P + C_1'C_1 = 0$$

$$A - B_2(D_{12}'D_{12}')^{-1}(B_2'P + D_{12}'C_1) + \frac{B_1B_1'}{g^2}P \text{ stable}$$



$$F_\infty = -(D_{12}'D_{12}')^{-1}(B_2'P + D_{12}'C_1)$$

$Q(s)$  is a stable system, proper and satisfying  $\|Q(s)\|_\infty < \gamma$ .

## Comments

- As  $(\gamma \rightarrow \infty)$  the central controller ( $Q(s)=0$ ) coincides with the  $H_2$  optimal controller
- The parametrized family directly includes the only static controller  $u=F_\infty x$ .
- The proof of the previous result, in its general fashion, would require too much time. Indeed, the necessary part requires a digression on the Hankel-Toeplitz operator. If we drop off the result on the parametrization and limit ourselves to the static state-feedback problem, i.e.  $u=Kx$ , the following simple result holds:

### THEOREM 6

There exists a stabilizing control law  $u=Kx$  such that the norm of the system  $(A+B_2K, B_1, C_1+D_{12}K)$  is less than  $\gamma$  if and only if there exists a positive definite solution of the inequality:

$$A'P + PA - (PB_2 + C'D_{12})(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1) + P \frac{B_1B_1'}{g^2} P + C_1'C_1 < 0$$



## Observation: LMI

$$P > 0$$

$$A'P + PA - (PB_2 + C'D_{12})(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1) \\ + P \frac{B_1B_1'}{\mathbf{g}^2} P + C_1'C_1 < 0$$



*Schur Lemma*  
 $X = \mathbf{g}^2 P^{-1}$

$$X > 0$$

$$\begin{bmatrix} -XA_c' - A_cX - B_1B_1' + \mathbf{g}^2 B_2B_2' & XC_{1c}' \\ C_{1c}X & \mathbf{g}^2 I \end{bmatrix} > 0$$

$$A_c = A - B_2(D_{12}'D_{12})^{-1}D_{12}'C_1$$

$$C_{1c} = (I - D_{12}(D_{12}'D_{12})^{-1}D_{12}')C_1$$

## Proof of Theorem 6

Assume that there exists  $K$  such that  $A+B_2K$  is stable and the norm of the system  $(A+B_2K, B_1, C_1+D_{12}K)$  is less than  $\gamma$ . Then, from the we know that there exists a positive definite solution of the inequality

$$(A + B_2K)'P + P(A + B_2K) + \mathbf{g}^{-2}PB_1B_1'P + (C_1 + D_{12}K)'(C_1 + D_{12}K) < 0 \quad (*)$$

Now, defining

$$F = -(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1)$$

the Riccati inequality can be equivalently rewritten as

$$A'P + PA - (PB_2 + C'D_{12})(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1) + P\frac{B_1B_1'}{\mathbf{g}^2}P + C_1'C_1 + (K - F)'(K - F) < 0$$

so concluding the proof. Vice-versa, assume that there exists a positive definite solution of the inequality. Then, inequality (\*) is satisfied with  $K=F$ , so that with such a  $K$ , the norm of the closed-loop transfer function is less than  $\gamma$  (BRL).

# Parametrization of all algebraic “state-feedback” controllers

Define:

$$A_R = [A \quad B_2]$$

$$C_R = [C_1 \quad D_{12}]$$

$$W = [W_1 \quad W_2]$$

## **THEOREM 7**

The set of all controllers  $u=Kx$  such that the  $H_\infty$  norm of the closed-loop system is less than  $\gamma$  is given by:

$$K = W_2' W_1^{-1}$$

$$W_1 > 0$$

$$\begin{bmatrix} -A_R W - W A_R' - B_1 B_1' & W C_R' \\ C_R W' & \mathbf{g}^2 I \end{bmatrix} > 0$$

## Proof of Theorem 7

We prove the theorem in the simple case in which  $C_1'D_{12}=0$  and  $D_{12}'D_{12}=I$ . The LMI is equivalent to

$$K = W_2'W_1^{-1}$$

$$W_1 > 0$$

$$AW_1 + W_1A' + B_1B_1' + W_2B_2' + B_2W_2' + \mathbf{g}^{-2}W_1C_1'C_1W_1 + \mathbf{g}^{-2}W_2W_2' < 0$$

If there exists  $W$  satisfying such an inequality, it follows that the inequality

$$P(A + B_2K) + (A + B_2K)'P + \mathbf{g}^{-2}PB_1B_1'P + C_1'C_1 + KK' < 0$$

Is satisfied with  $K=W_2'W_1^{-1}$  and  $P=\gamma^2W_1^{-1}$ . The result follows from the BRL. Vice-versa, assume that there exists  $P>0$  satisfying this last inequality. The conclusion directly follows by letting  $W_1=\gamma^2P^{-1}$  and  $W_2=W_1K'$ .

## Example

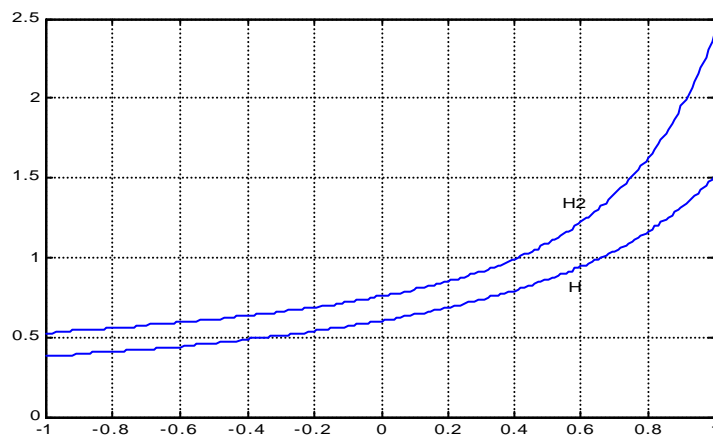
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (w + u)$$

$$z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\text{robustness } \Delta A = \begin{bmatrix} 0 & 0 \\ \Omega & 0 \end{bmatrix}$$

$$\gamma_{\text{inf}} = 0.75, \quad \gamma = 0.76$$

$$\|T(z_1, w)\|_{\infty}$$



$\Omega$

## Mixed Problem $H_2/H_\infty$

$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{12} u$$

$$\mathbf{x} = Lx + Mu$$

The problem consists in finding  $u=Kx$  in such a way that

$$\min_K \|T(\mathbf{x}, w, s, K)\|_2 : \|T(z, w, s, K)\|_\infty \leq \mathbf{g}$$

- For instance, take the case  $z=\xi$ . The problem makes sense since a blind minimization of the  $H_\infty$  infinity norm may bring to a serious deterioration of the  $H_2$  (mean squares) performances.
- Obviously, the problem is non trivial only if the value of  $\gamma$  is included in the interval  $(\gamma_{\text{inf}} \gamma_2)$ , where  $\gamma_{\text{inf}}$  is the infimum achievable  $H_\infty$  norm for  $T(z,w,s,K)$  as a function of  $K$ , whereas  $\gamma_2$  is the  $H_\infty$  norm of  $T(z,w,s,K_{\text{OTT}})$  where  $K_{\text{OTT}}$  is the optimal unconstrained  $H_2$  controller.
- This problem has been tackled in different ways, but an analytic solution is not available yet. In the literature, many sub-optimal solutions can be found (Nash-game approach, convex optimization, inverse problem, ...)

## Post-Optimization procedure

$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{12} u$$

$$\mathbf{x} = Lx + Mu$$

simplifying assumptions  $L'M=0$ ,  $M'M=I$

Let  $K_{sub}$  a controller such that  $\|T(z,w,s,K_{sub})\|_{\infty} < \gamma$ . For  $\alpha \in [0, 2]$ , define the matrices

$$A_{\alpha} = A + (1-\alpha)^2 B_2 K_{sub}$$

$$B_{2\alpha} = (2\alpha - \alpha^2)^{1/2} B_2$$

$$C_{\alpha} = L + (1-\alpha) M K_{sub}$$

And the standard Riccati equation

$$A_a' P_a + P_a A_a - P_a B_{2a} B_{2a}' P_a + C_a' C_a = 0 \quad (\mathbf{a} - RIC)$$

Notice that if  $(A,L)$  is detectable and  $(A,B_2)$  stabilizable, such an equation always admits the stabilizing solution (positive semidefinite)  $P_{\alpha}$  for each  $\alpha \in [0, 2]$ . For each  $\alpha \in [0, 2]$  we can write the family of controllers

$$K_a = (1-a) K_{sub} - a B_2' P_a \quad (\mathbf{a} - con)$$

# Post-Opt Algorithm

## THEOREM 8

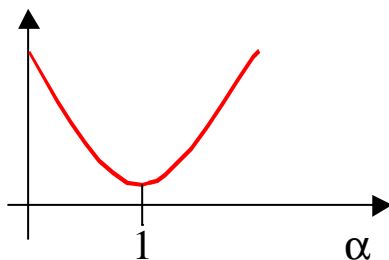
Each controller  $K_a$  of the family ( $\alpha$ -con) is a stabilizing controller and is such that

$$\|T(\mathbf{x}, w, s, K_a)\|_2 = \|T(\mathbf{x}, w, s, K_{2-a})\|_2$$

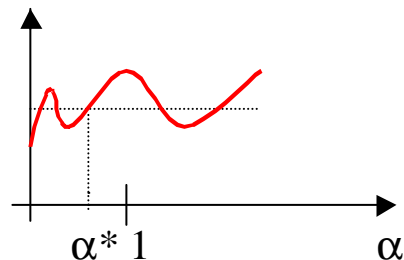
$$(1-a) \frac{d}{da} \|T(\mathbf{x}, w, s, K_a)\|_2 \leq 0$$

**Interpretation:** For  $\alpha=0$ , the controller is  $K_0=K_{\text{sub}}$ . Hence,  $\|T(z,w,s,K_0)\|_\infty < \gamma$ . For  $\alpha=1$ , the controller is  $K_1=K_{\text{OTT}}$ , so it coincides with the optimal unconstrained  $H_2$  controller, hence  $\|T(z,w,s,K_1)\|_\infty \geq \gamma$ . One varies  $\alpha$  till the value  $\alpha^*$  which is closest to  $\alpha=1$  and such that  $\|T(z,w,s,K_{\alpha^*})\|_\infty = \gamma$ . The reason of the equality is in the fact (as it is possible to proof starting from the necessary conditions) that the optimal mixed controller  $K_{\text{ottmix}}$  satisfies  $\|T(z,w,s,K_{\text{ottmix}})\|_\infty = \gamma$ .

$\|T(z,w,s,K_\alpha)\|_2$



$\|T(z,w,s,K_\alpha)\|_\infty$





## Proof of Theorem 8

First notice that the equation ( $\alpha$ -RIC) can be rewritten as:

$$(A + B_2 K_a)' P_a + P_a (A + B_2 K_a) + K_a' K_a + L' L = 0 \quad (\mathbf{a} - RIC)$$

so that

$$\|T(\xi, w, s, K_\alpha)\|_2 = (\text{Trace}(B_1' P_\alpha B_1))^{1/2}$$

The fact that the norm  $\|T(\xi, w, s, K_\alpha)\|_2$  is symmetric with respect to  $\alpha=1$  follows directly ( $\alpha$ -RIC) by inspection. Taking the derivative with respect to  $\alpha$  one obtains

$$F_a' X_a + X_a F_a - 2(1 - \mathbf{a}) \Lambda_a' \Lambda_a = 0$$

where  $X_\alpha$  is the derivative of  $P_\alpha$  with respect to  $\alpha$  and

$$F_\alpha = A_\alpha - B_{2\alpha} B_{2\alpha}' P_\alpha, \quad \Lambda_\alpha = K_{\text{sub}} + P_\alpha B_2$$

Matrix  $F_\alpha$  is stable since  $P_\alpha$  is the stabilizing solution of ( $\alpha$ -RIC). Hence, for the Lyapunov Lemma the conclusion follows.

## Example

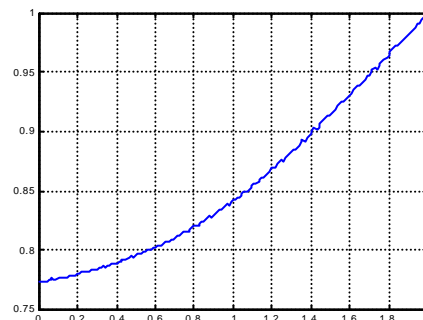
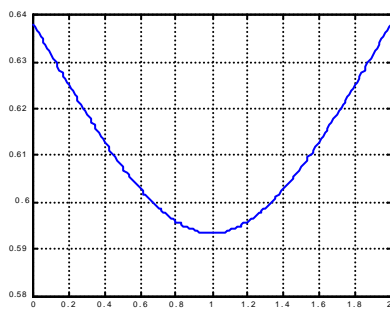
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (w + u)$$

$$z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

robustness  $\Delta A = \begin{bmatrix} 0 & 0 \\ \Omega & 0 \end{bmatrix}$

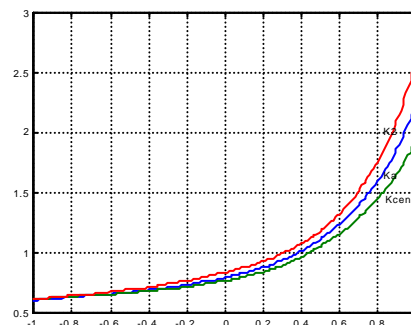
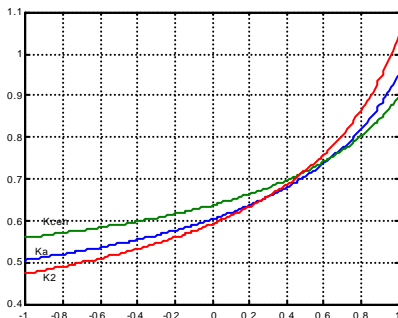
$$\gamma_{\text{inf}} = 0.75, \quad \gamma_2 = 0.8415, \quad \gamma = 0.8$$

$$K_{\text{sub}} = K_{\text{cen}} = [-0.6019 \quad -0.7676]$$



$$a^* = 0.56, \quad K_{a^*} = [-0.4981 \quad -0.5424]$$

## Performances with respect to W



# Disturbance Feedforward

$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{12} u$$

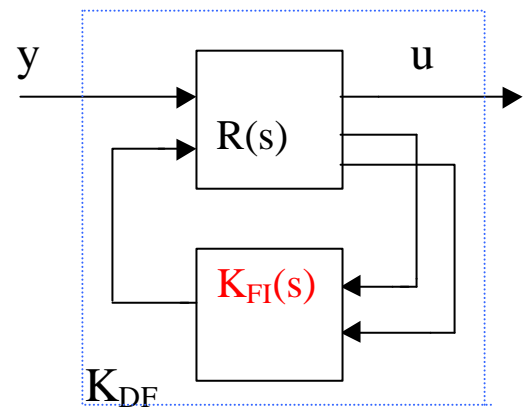
$$y = C_2 x + w$$

Same assumptions as FI+stability of  $A-B_1 C_2$

$C_2=0 \rightarrow$  direct compensation

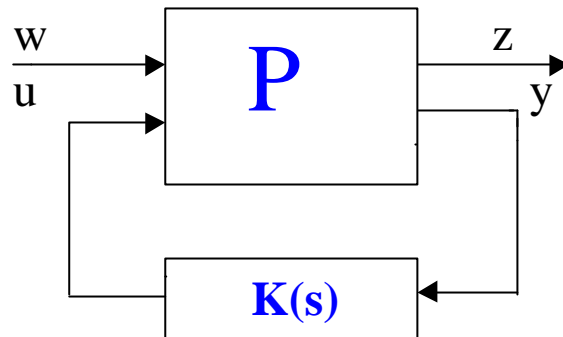
$C_2 \neq 0 \rightarrow$  indirect compensation

$$R(s) = \left[ \begin{array}{c|cc} A-B_1 C_2 & B_1 & B_2 \\ \hline 0 & 0 & I \\ I & 0 & 0 \\ -C_2 & I & 0 \end{array} \right]$$



The proof of the DF theorem consist in verifying that the transfer function from  $w$  to  $z$  achieved by the FI regulator for the system  $A, B_1, B_2, C_1, D_{12}$  is the same as the transfer function from  $w$  to  $z$  obtained by using the regulator  $K_{DF}$  shown in the figure.

# Output Estimation

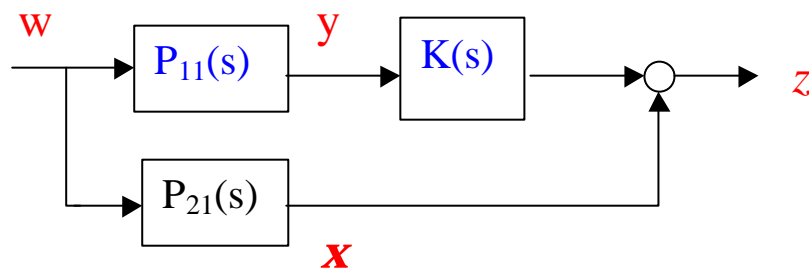


$$\dot{x} = Ax + B_1 w + (B_2 u)$$

$$z = C_1 x + u$$

$$u = -\hat{x}$$

$$y = C_2 x + D_{21} w$$



## Assumptions

$(A, C_2)$  detectable

$$D_{21} D_{21}' > 0$$

$(A_f, B_{1f})$  stabilizable

$(A - B_2 C_1)$  stable

$$A_f = A - B_1 D_{21}' (D_{21} D_{21}')^{-1} C_2$$

$$B_{1f} = B_1 (I - D_{21}' (D_{21} D_{21}')^{-1} D_{21})$$

# Solution of the Output Estimation problem

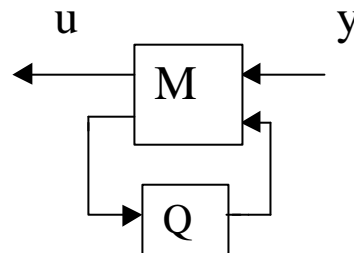
## THEOREM 9

There exists a feasible controller (Filter) such that  $\|T(z,w,s)\|_\infty < \gamma$  if and only if there exists the stabilizing positive semidefinite solution of the Riccati equation

$$A\Pi + \Pi A' - (\Pi C_2' + B_1 D_{21}') (D_{21} D_{21}')^{-1} (C_2 \Pi + D_{21} B_1) + \Pi \frac{C_1' C_1}{g^2} \Pi + B_1 B_1' = 0$$

$$A - (\Pi C_2' + B_1 D_{21}') (D_{21} D_{21}')^{-1} C_2 \Pi \frac{C_1' C_1}{g^2} \text{ stable}$$

$$M(s) = \left[ \begin{array}{c|cc} A_{ff} & L_\infty & -B_2 \gamma^{-2} \Pi C_1' \\ \hline C_1 & 0 & I \\ C_{2f} & I_f & 0 \end{array} \right]$$



$$A_{ff} = A - \Pi C_2' (D_{21} D_{21}')^{-1} C_2 - B_2 C_1$$

$$C_{2f} = (D_{21} D_{21}')^{-1} C_2$$

$$I_f = (D_{21} D_{21}')^{-1}$$

$$L_\infty = -(\Pi C_2' + B_1 D_{21}') (D_{21} D_{21}')^{-1}$$

$Q(s)$  is a proper stable system satisfying  $\|Q(s)\|_\infty < \gamma$ .

## Proof of Theorem 9

It is enough to observe that the “transpose system”

$$\dot{\mathbf{I}} = \mathbf{A}'\mathbf{I} + \mathbf{C}'_1\mathbf{V} + \mathbf{C}'_2\mathbf{n}$$

$$\mathbf{m} = \mathbf{B}'_1\mathbf{I} + \mathbf{D}'_{21}\mathbf{n}$$

$$\mathbf{J} = \mathbf{B}'_2\mathbf{I} + \mathbf{V}$$

Has the same structure as the system defining the DF problem. Hence the solution is the “transpose” solution of the DF problem. It is worth noticing that the solution depends on the particular linear combination of the state that one wants to estimate.

## Example

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 + \Omega \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_1$$

$$y = [1 \quad 0]x + w_2$$

damping  $\alpha = (1 - \Omega)/2$

Let consider six filters:

**Filter K1:** Kalman filter. The noises are assumed to be white uncorrelated gaussian noises with identity intensities, namely  $W_1=1, W_2=1$ .

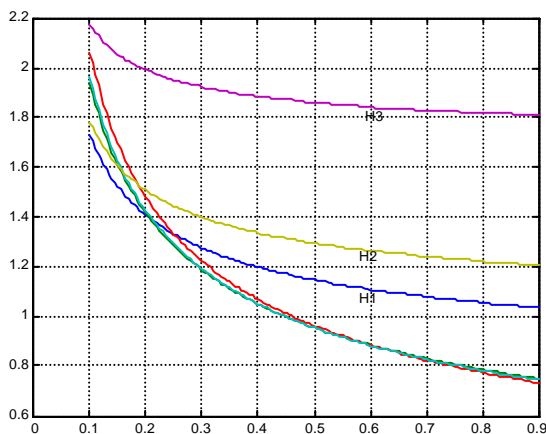
**Filter K2:** Kalman filter. The noises are assumed to be white uncorrelated gaussian noises with  $W_1=1, W_2=0.5$ .

**Filter K3:** Kalman filter. The noises are assumed to be white uncorrelated gaussian noises with  $W_1=0.5, W_2=1$ .

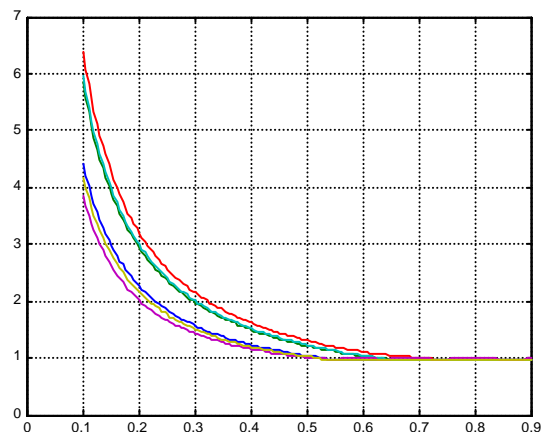
**Filters H1,H2,H3:**  $H_\infty$  filters with  $\gamma=1.1, \gamma=1.01, \gamma=1.005$  (notice that with  $\gamma=1$  the stabilizing solution of the Riccati equation does not exist).

Other techniques of **Robust filtering** are possible

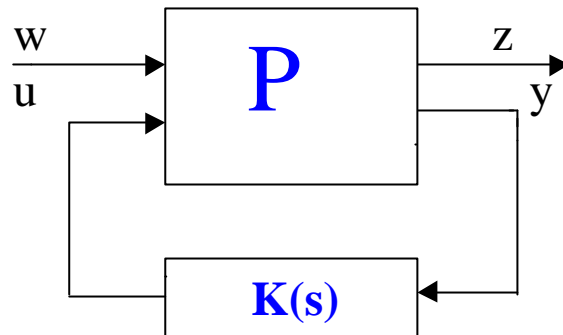
Norm  $H_2$



Norm  $H_\infty$



# Partial Information



$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{12} u$$

$$y = C_2 x + D_{21} w$$

Assumptions: FI + DF.

## Theorem 10

There exists a feasible such that  $\|T(z,w,s)\|_\infty < \gamma$  if and only if

- There exists the stabilizing positive semidefinite solution of

$$A'P + PA - (PB_2 + C_1'D_{12})(D_{12}'D_{12})^{-1}(B_2'P + D_{12}'C_1) + P\frac{B_1B_1'}{g^2}P + C_1'C_1 = 0$$

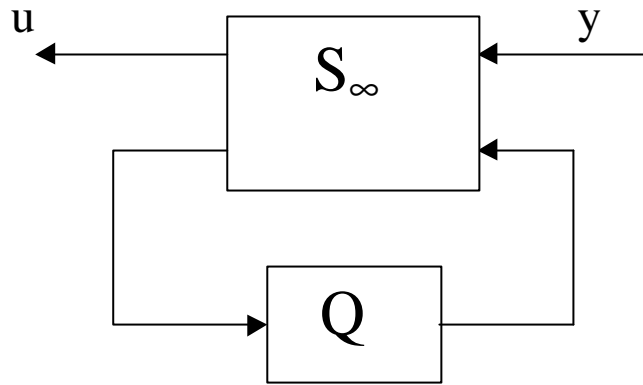
- There exists the stabilizing positive semidefinite solution of

$$A\Pi + \Pi A' - (\Pi C_2' + B_1 D_{21}') (D_{21} D_{21}')^{-1} (C_2 \Pi + D_{21} B_1') + \Pi \frac{C_1' C_1}{g^2} \Pi + B_1 B_1' = 0$$

- $\max_{\gamma} I_{\gamma}(P\Pi) < g^2$



## Structure of the regulator



$$S_\infty(s) = \left[ \begin{array}{c|cc} A_{\text{fin}} & B_{1\text{fin}} & B_{2\text{fin}} \\ \hline F_\infty & 0 & (D_{21}D_{21}')^{-1/2} \\ C_{2\text{fin}} & (D_{21}D_{21}')^{-1/2} & 0 \end{array} \right]$$

$$A_{\text{fin}} = A - B_2(D_{12}'D_{12})^{-1}D_{12}C_1 - B_2(D_{12}'D_{12})^{-1}B_2'P + \gamma^2 B_1 B_1' P + (I - \gamma^2 \Pi P)^{-1} L_\infty (C_2 + \gamma^2 D_{21} B_1' P)$$

$$B_{1\text{fin}} = -(I - \gamma^2 \Pi P)^{-1} L_\infty$$

$$B_{2\text{fin}} = (I - \gamma^2 \Pi P)^{-1} (B_2 + \gamma^2 \Pi C_1' D_{12}) (D_{12}' D_{12})^{-1/2}$$

$$C_{2\text{fin}} = -(D_{21} D_{21}')^{-1} (C_2 + \gamma^2 D_{21} B_1' P)$$

## Comments

It is easy to check that the central controller ( $Q(s)=0$ ) is described by:

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B_2u + Z_\infty L_\infty (C_2\mathbf{x} - y + D_{21}w^*) + B_1w^* \\ u &= F_\infty\mathbf{x}\end{aligned}$$

This closely resembles the structure of the optimal  $H_2$  controller. Notice, however, that the well-known separation principle does not hold for the presence of the worst disturbance  $w^* = B_1' P \xi$ .

## Proof of Theorem 10 (sketch)

Define the variables  $r$  e  $q$  as:

$$w = r + \gamma^{-2} B_1' P x$$

$$q = u - F_\infty x$$

In this way, the system becomes

$$\dot{x} = (A + \mathbf{g}^{-2} B_1 B_1' P) x + B_1 r + B_2 u$$

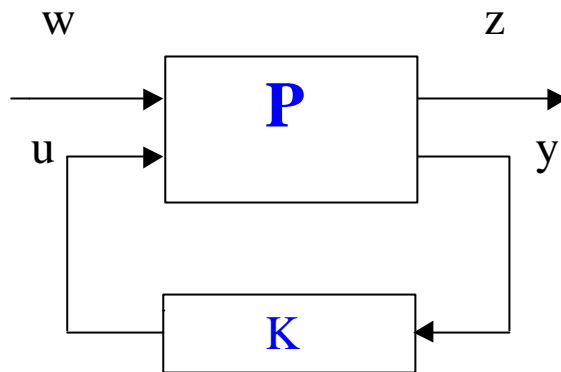
$$q = -F_\infty x + u$$

$$y = (C_2 + \mathbf{g}^{-2} D_{21} B_1') x + D_{21} w$$

This system has the same structure as the one of the OE problem. Hence, the solution can be obtained from the solution of the OE problem, by recognizing that the solution  $\Pi_t$  of the relevant Riccati equation is related to the solutions  $P$  e  $\Pi$  above in the following way:

$$\Pi_t = \Pi (I - \mathbf{g}^{-2} P \Pi)^{-1}$$

# Operatorial Approach



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

$$T(z, w, s) = P_{11}(s) + P_{12}(s)(I - K(s)P_{22}(s))^{-1}K(s)P_{21}(s)$$

$$T(z, w, s) = T_1(s) - T_2(s)Q(s)T_3(s)$$

The functions  $T_1$ ,  $T_2$ ,  $T_3$  depend on a double coprime factorization of  $P_{22}$ . The function  $Q(s)$  is the Youla-Kucera parameter (stable and proper function).

- Inner-outer factorization and reduction of the problem to a distance Nehari problem

## Laurent and Hankel operators

- Let  $F(s) \in L_\infty$ . The map  $\Lambda_F: L_2 \rightarrow L_2$  defined as

$$\Lambda_F : G(s) \rightarrow \Lambda_F G(s) = F(s)G(s)$$

is called the Laurent operator (with symbol F).

- Consider  $G(s) \in L_2$  and let  $G(s) = G_s(s) + G_a(s)$ , with  $G_s(s) \in H_2$  and  $G_a(s) \in H_2^\perp$ . The map  $\Pi_s: L_2 \rightarrow H_2$  defined as

$$\Pi_s : G(s) \rightarrow \Pi_s G(s) = G_s(s)$$

is called the stable projection.

- Consider  $G(s) \in L_2$  and let  $G(s) = G_s(s) + G_a(s)$ , with  $G_s(s) \in H_2$  and  $G_a(s) \in H_2^\perp$ . The map  $\Pi_a: L_2 \rightarrow H_2^\perp$  defined as

$$\Pi_a : G(s) \rightarrow \Pi_a G(s) = G_a(s)$$

is called the unstable projection.

- Let  $F(s) \in L_\infty$ . The map  $\Gamma_F: H_2^\perp \rightarrow H_2$  defined as

$$\Gamma_F : G(s) \rightarrow \Gamma_F G(s) = \Pi_s \Lambda_F G(s)$$

is called the Hankel operator (with symbol F).

Notice that  $\Lambda_F$  is linear and  $\|\Lambda_F\| = \|F(s)\|_\infty$  so that is a bounded operator. It is immediate to check that the projections  $\Pi_s$  and  $\Pi_a$  are indeed linear operators. The Hankel operator is the result of the composition of the Laurent operator and the stable projection, i.e.  $\Gamma_F = \Pi_s \Lambda_F$ . Notice also that only the stable part of  $F(s)$  contributes to  $\Gamma_F G(s)$ . Indeed, since  $G(s) \in H_2^\perp$  it follows  $\Gamma_F G(s) = \Pi_s \Lambda_F G(s) = \Pi_s F(s)G(s) = \Pi_s [(F_s(s) + F_a(s))G(s)] = \Pi_s F_s(s)G(s) = \Gamma_{F_s} G(s)$

## The adjoint operators

- The adjoint Laurent operator (with symbol F) is the adjoint operator with symbol  $F^\sim$ , i.e.

$$\Lambda_{F^\sim}^* = \Lambda_F$$

- The adjoint Hankel operator (with symbol F) is

$$\Gamma_{F^\sim}^* = \Pi_a \Lambda_F^*$$

**Laurent:**  $G_1(s) \in L_2$  and  $G_2(s) \in L_2$ . Then

$$\langle G_1, \Lambda_{F^\sim}^* G_2 \rangle = \langle \Lambda_F G_1, G_2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{trace}[G_1(-j\omega)' F(-j\omega)' G_2(j\omega)] d\omega =$$

$$\langle G_1, F^\sim G_2 \rangle = \langle G_1, \Lambda_{F^\sim} G_2 \rangle$$

**Hankel:**  $H(s) \in H_2$  and  $G(s) \in H_2^\perp$ . Recall that  $\langle a, b \rangle = 0$  if  $a \in H_2$  and  $b \in H_2^\perp$ . Then

$$\langle G, \Gamma_{F^\sim}^* H \rangle = \langle \Gamma_F G, H \rangle = \langle \Pi_s F G, H \rangle = \langle F G, H \rangle =$$

$$\langle G, F^\sim H \rangle = \langle G, \Pi_a F^\sim H \rangle = \langle G, \Pi_a \Lambda_{F^\sim} H \rangle$$

# Time-domain interpretation of the Hankel operator

Let  $F(s) = C(sI - A)^{-1}B$ , with  $A$  stable,  $(A, C)$  observable and  $(A, B)$  reachable. Hence, the inverse Laplace transform of  $F(s)$  is

$$f(t) = \begin{cases} 0 & t \leq 0 \\ Ce^{At}B & t > 0 \end{cases}$$

Now, consider the map  $\Gamma_f : L_2(-\infty, 0) \rightarrow L_2(0, +\infty)$  Defined by

$$\Gamma_f : u(t) \rightarrow \Gamma_f u(t) = y(t) = \begin{cases} 0 & t \leq 0 \\ Ce^{At} \int_{-\infty}^0 e^{-At} Bu(t) dt & t > 0 \end{cases}$$

*This operator maps past input to future outputs. Indeed, the past inputs (with  $x(-\infty) = 0$ ) contribute to form the state  $x(0)$  from which the free output can be calculated.  $\Gamma_f$  is the time-domain counterpart of  $G$ .  $\Gamma_f$  can be seen as the composition of two operators  $\Psi_o$  (the observability operator) and  $\Psi_r$  (the reachability operator)*

$$\Gamma_f = \Psi_r \Psi_o, \quad \Psi_r : L_2(-\infty, 0) \rightarrow R^n, \quad \Psi_o : R^n \rightarrow L_2(0, \infty)$$

$$\Psi_r : u(t) \rightarrow \Psi_r u(t) := x = \int_{-\infty}^0 e^{-At} Bu(t) dt$$

$$\Psi_o : x \rightarrow \Psi_o x := y(t) = \begin{cases} 0 & t < 0 \\ Ce^{At} x & t \geq 0 \end{cases}$$

*The operator  $\Psi_o$  is injective thanks to the observability of  $(A, C)$ . Indeed, from  $\Psi_o x = 0$  it follows  $x = 0$ .*

*Conversely, the operator  $\Psi_r$  is surjective thanks to the reachability of  $(A, B)$ . Indeed, given any point  $x$  of  $R^n$  it follows that  $\Psi_r u(t) = x$ , where  $u(t) = 0$ , for  $t > 0$  and  $u(t) = B e^{-At} P_r^{-1}$  for  $t \leq 0$ , where  $P_r$  is the unique solution of the Lyapunov equation  $A P_r + P_r A + B B^T = 0$*

# Time-domain interpretation of the adjoint Hankel operator

The adjoint reachability and observability operators are as follows:

$$\Psi_r^* : x \rightarrow \Psi_r^* x = u(t) = \begin{cases} B' e^{-A't} x & t \leq 0 \\ 0 & t > 0 \end{cases}$$

$$\Psi_o^* : y(t) \rightarrow \Psi_o^* y(t) = x = \int_0^{\infty} e^{-A't} C' y(t) dt$$

Therefore the adjoint Hankel operator in time-domain is

$$\Gamma_f^* : L_2[0, \infty) \rightarrow L_2(-\infty, 0]$$

$$\Gamma_f^* : y(t) \rightarrow \Gamma_f^* y(t) = u(t) = \begin{cases} B' e^{A't} \int_0^{\infty} e^{-A't} C' y(t) dt & t \leq 0 \\ 0 & t > 0 \end{cases}$$

The rank of  $\Psi_r$  and  $\Psi_o$  is  $n = \text{McMillan degree of } C(sI-A)^{-1}B$ . Moreover, since  $\Psi_o$  is injective and  $\Psi_r$  surjective, it turns out that  $\Gamma_f$  (and  $\Gamma_F$ ) has rank  $n$ . Therefore the self adjoint operator  $\Gamma_F^* \Gamma_F$  has rank  $n$ , so that it admits eigenvalues.

## Reachability and observability grammians

$$\Psi_r \Psi_r^* = \int_0^{\infty} e^{At} B B' e^{A't} dt = P_r$$

$$\Psi_o^* \Psi_o = \int_0^{\infty} e^{A't} C' C e^{At} dt = P_o$$

Notice that  $P_r$  and  $P_o$  are the unique (positive definite) solutions of the Lyapunov equations:

$$AP_r + P_r A' + BB' = 0$$

$$P_o A + A' P_o + C' C = 0$$

The operator  $\Gamma_F^* \Gamma_F$  and the matrix  $P_r P_o$  share the same nonzero eigenvalues. To see this notice that  $\Gamma_f^* \Gamma_f = \Psi_r^* \Psi_o^* \Psi_o \Psi_r = P_r P_o$

The Hankel norm of the system is

$$\|\Gamma_F\| = \|\Gamma_f\| = \sqrt{\mathbf{I}_{\max}(P_r P_o)}$$

Example:

$$F(s) = \frac{5(s-1)(s+5)(s-3)}{(s+1)^2(s-7)} = 5 + \frac{22.5}{s-7} + \frac{2.5(3s-5)}{(s+1)^2}$$

Consider the stable part and take a minimal realization:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [-12.5 \quad 7.5]$$

It results

$$P_r = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}, P_o = \begin{bmatrix} 303.125 & 78.125 \\ 78.125 & 53.125 \end{bmatrix}, \mathbf{I}_{\max}(P_r P_o) = 81.3827, \|\Gamma_F\| = 9.0212$$





## Scalar case

It is possible to perform an inner-outer factorization of  $T_4 = T_2 T_3 = T_{4i} T_{4o}$  so that, letting  $F = T_{4i}^{-1} T_1$  and  $X = T_{4o} Q$  we have:

$$\begin{aligned} \|T_1(s) - T_2(s)Q(s)T_3(s)\|_\infty &= \|T_1(s) - T_4(s)Q(s)\|_\infty = \\ \|T_{4i}(s)^{-1}T_1(s) - T_{4o}(s)Q(s)\|_\infty &= \|F(s) - X(s)\|_\infty \end{aligned}$$

The problem is then reduced to a Nehari problem.

Now, let

$$F(s) = F_a(s) + F_s(s) + D, \quad F_a(s) \in H_2^\perp, \quad F_s(s) \in H_2$$

$$F_a(s) = C(sI - A)^{-1} B$$

$$A'P + PA = BB', \quad AQ + QA' = C'C$$

$$PQ\mathbf{b} = \mathbf{l}_{\max}(PQ)\mathbf{b}, \quad \mathbf{c} = [\mathbf{l}_{\max}(PQ)]^{-1} P\mathbf{b}$$

$$f(s) = B'(sI - A')^{-1} \mathbf{b}, \quad g(s) = C(sI - A)^{-1} \mathbf{c}$$

### Theorem 11

The function

$$X^0(s) = F(s) - \frac{\mathbf{l}_{\max}(PQ)g(s)}{f(s)}$$

is such that

$$\|F(s) - X^0(s)\| = \inf_{x(s) \in H_\infty} \|F(s) - X(s)\|$$

## Proof of Theorem 11

From  $F(s)$  construct  $f(s)$ ,  $g(s)$  and  $\lambda_{\max}(PQ)$ . Let be  $X^o(s)$  be the optimal solution and define  $h(s)=(F(s)-X^o(s))f(s)$ . Notice that  $h(s) \in L_2$ , since  $F(s)-X^o(s) \in L_\infty$  and  $f(s) \in H_2$ . It follows:

$$\begin{aligned} \left\| h(s) - \Gamma_{F \sim}^* f(s) \right\|_2^2 &= \langle h(s) - \Gamma_{F \sim}^* f(s), h(s) - \Gamma_{F \sim}^* f(s) \rangle = \\ &\langle h(s), h(s) \rangle + \langle \Gamma_{F \sim}^* f(s), \Gamma_{F \sim}^* f(s) \rangle - 2\langle h(s), \Gamma_{F \sim}^* f(s) \rangle = \end{aligned}$$

Now, taking into account that  $\Gamma_{F \sim}^* f(s) \in H_2^\perp$  and  $X^o f(s) \in H_2$ , it follows

$$\begin{aligned} \langle h(s), \Gamma_{F \sim}^* f(s) \rangle &= \langle \Pi_a [F(s) - X^o(s)] f(s), \Gamma_{F \sim}^* f(s) \rangle = \\ &\langle \Pi_a F(s) f(s), \Gamma_{F \sim}^* f(s) \rangle = \langle \Gamma_{F \sim}^* f(s), \Gamma_{F \sim}^* f(s) \rangle \end{aligned}$$

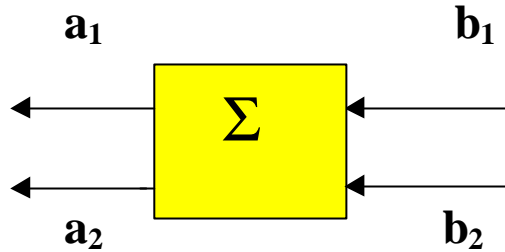
So that

$$\begin{aligned} \left\| h(s) - \Gamma_{F \sim}^* f(s) \right\|_2^2 &= \langle h(s), h(s) \rangle - \langle \Gamma_{F \sim}^* f(s), \Gamma_{F \sim}^* f(s) \rangle = \\ &\left\| (F(s) - X^o(s)) f(s) \right\|_2^2 - \mathbf{I}_{\max}(PQ) \|f(s)\|_2^2 \leq \\ &\left[ \left\| (F(s) - X^o(s)) \right\|_\infty^2 - \mathbf{I}_{\max}(PQ) \right] \|f(s)\|_2^2 \end{aligned}$$

The Nehari theorem says that the minimum of  $\|F(s)-X(s)\|_\infty$  is equal to  $\|\Gamma_{F \sim}\| = \lambda_{\max}(PQ)$  [the proof is omitted]. Hence, it follows  $\|F(s)-X^o(s)\|_\infty = \lambda_{\max}(PQ)$  so that  $h(s) = \Gamma_{F \sim}^* f(s)$ . On the other hand, as it is easy to check,  $\Gamma_{F \sim}^* f(s) = g(s)$ . Therefore,

$$X^o(s) = F(s) - \|\Gamma_{F \sim}\| \frac{g(s)}{f(s)}$$

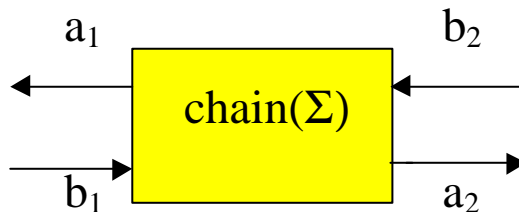
# Chain Scattering Representation



$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \Sigma \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

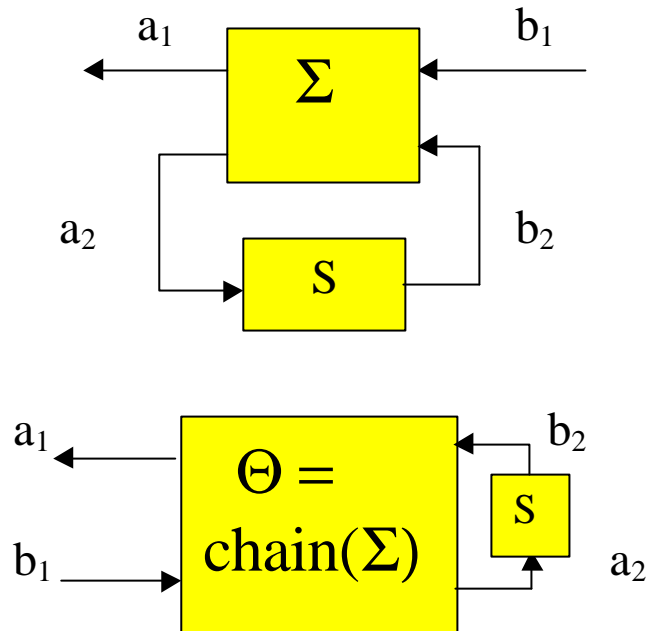
If  $\Sigma_{21}$  is invertible, then  $b_1 = \Sigma_{21}^{-1}(a_2 - \Sigma_{22}b_2)$  and

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \text{Chain}(\Sigma) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{Chain}(\Sigma) = \begin{bmatrix} \Sigma_{12} & -\Sigma_{11}\Sigma_{21}^{-1}\Sigma_{22} & \Sigma_{11}\Sigma_{21}^{-1} \\ -\Sigma_{21}^{-1}\Sigma_{22} & \Sigma_{21}^{-1} \end{bmatrix}$$



- Algebra of chain scattering representation

# Homographic Transformation



$$a_1 = \Phi b_1, \quad \Phi = LF(\Sigma; S) = \Sigma_{11} + \Sigma_{12}S(I - \Sigma_{22}S)^{-1}\Sigma_{21}$$

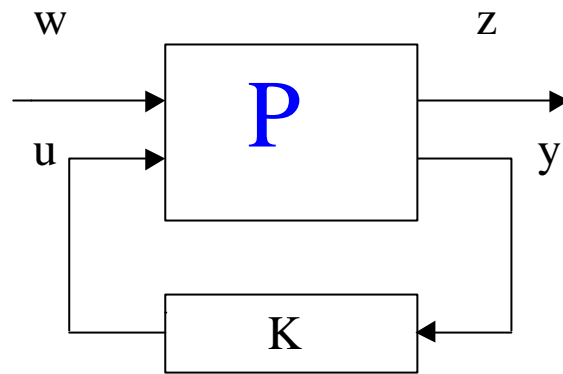
$$\Phi = HM(\Theta; S) = (\Theta_{11}S + \Theta_{12})(\Theta_{21}S + \Theta_{22})^{-1}$$

**Y is J-unitary**

$$\Psi^{\sim}(s)J\Psi(s) = J,$$

$$\Psi^{\sim}(s) = \Psi(-s)', \quad J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

# The Standard Problem via chain-scattering representation



We say that  $G(s)$  admits a J-unitary factorization if

$$G(s) = \Theta(s)\Lambda(s)$$

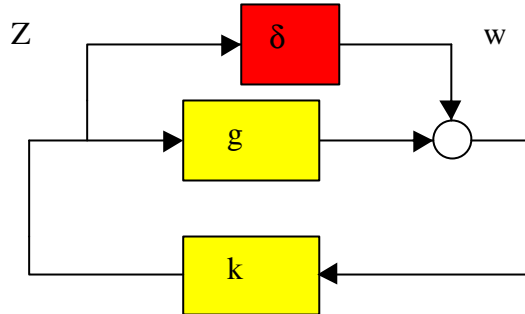
Where  $\Theta(s)$  is J-unitary and  $\Lambda(s)$  is unimodular (stable with stable inverse) .

## Theorem 12

Assume that  $P$  has a chain scattering representation  $G = \text{chain}(P)$  such that  $G$  is left invertible and does not have poles or zeros on the imaginary axis. Then, the  $H_\infty$  problem is solvable if and only if  $G$  has a J-unitary factorization  $G = \Theta\Lambda$ . The controllers  $K$  can be written as  $K = HM(\Lambda^{-1}; S)$  for each proper and stable  $S$ .

- **J-spectral Factorization in control and filtering**

# A robust stability problem



$$z = \frac{k}{1 - gk} w, \quad q = \frac{k}{1 - gk}$$

We want to maximize  $\|d\|_\infty$  (minimize  $\|q\|_\infty$ ). For the stability of the closed loop it must be

- (1) q stable
- (2) gq stable
- (3)  $(1+gq)g$  =stable

Hence, if  $p_i$  are the unstable poles of  $g$  (assumed to be simple and with  $\text{Re}(p_i) > 0$ )  $q$  must be such that

- (0) q minimum norm
- (1) q stable
- (2)  $q(p_i) = 0$
- (3)  $(1+gq)(p_i) = 0$

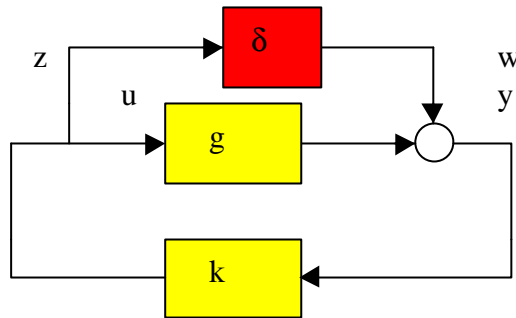
Letting

$$q^w = qa = q \prod_i \frac{s + \bar{p}_i}{p_i - s}$$

it follows that  $q^w$  must be such that

- (0)  $q^w$  minimum norm
- (1)  $q^w$  stable
- (2)  $q^w(p_i) = -ag^{-1}(p_i)$

## A simple example



where

$$g = \frac{s + 2}{(s + 1)(s - 1)}$$

**Find  $k$  in such a way that  $\|d\|_{\infty}$  is maximized.**

Take

$$a = \frac{s + 1}{1 - s}$$

and find  $q^w$  stable of minimum norm such that such that  $q^w(1) = -ag^{-1}(1) = 4/3$ . It follows  $q^w = 4/3$  so that  $\|d\|_{\infty \max} = 3/4$  and

$$k = \frac{q^w}{1 + q^w g} = -\frac{4(s + 1)}{(3s + 5)}$$



## In state-space

$$A=[0 \ 1; 1 \ 0], B_1=[0;0], B_2=[0;1], C_1=[0 \ 0], D_{12}=1, C_2=[2 \ 1], D_{21}=1.$$

Stabilizing solution of:

$$A'P+PA-PB_2B_2'P+PB_1B_1'P/\gamma^2+C_1'C_1=0$$

Stabilizing solution of:

$$AQ+QA'-PC_2'C_2Q+PC_1'C_1Q/\gamma^2+B_1B_1'=0$$

$$r_s(PQ) < \gamma^2$$

$P=[2 \ 2; 2 \ 2]$ , independently on  $\gamma$ .

$Q=[2/9 \ 2/9; 2/9 \ 2/9]$ , independently on  $\gamma$ .

$PQ=[8/9 \ 8/9; 8/9 \ 8/9]$  whose eigenvalues are 0 and 16/9. Hence  $\gamma_{\min}=4/3$  and  $\|d\|_{\infty \max} = 3/4$ .

Central controller.

Let

$$F=-B_2'P-D_{12}'C_1=-[2 \ 2]$$

$$L=-QC_2'-B_1D_{21}'=-[2 \ 2]'/3$$

$$Z=(I-QP/\gamma^2)^{-1}=[1-8\gamma^{-2}/9 \ -8\gamma^{-2}/9; -8\gamma^{-2}/9 \ 1-8\gamma^{-2}/9]/x, \quad x=(1-16\gamma^{-2}/9)^{-1}, \quad x \rightarrow 0$$

The central controller is:

$$K_{cen} = \begin{bmatrix} A + B_2F + \mathbf{g}^{-2}B_1B_1'P + ZLC_2 + ZLD_{21}B_1'P & -ZL \\ F & 0 \end{bmatrix}$$

whose transfer function is:

$$K_{cen} = \frac{-8(s+1)}{3x[s^2 + s(2 + \frac{6}{3x}) + 1 + \frac{10}{3x}]} = \frac{-8(s+1)}{3xs^2 + 6xs + 6s + 3x + 10} \rightarrow \frac{-4(s+1)}{3s+5}$$

## Operatorial approach

$$P(s) = \begin{bmatrix} 0 & 1 \\ 1 & g \end{bmatrix} \quad g = \frac{s+2}{s^2-1} = \frac{n}{m} \quad n = \frac{s+2}{(s+1)^2}, \quad m = \frac{s-1}{s+1}$$

Bezout

$$-ny + mx = 1, \quad x = \frac{s+5/3}{s+1}, \quad y = -4/3$$

Model Matching

$$t_1 = p_{11} + p_{12}myp_{21} = \frac{-4/3(s-1)}{s+1}$$

$$t_2 = t_3 = \frac{s-1}{s+1}$$

Inner-outer factorization of  $t_4=t_2t_3$

$$t_4 = t_{4i}t_{4o}, \quad t_{4i} = t_4 = \frac{(s-1)^2}{(s+1)^2}, \quad t_{4o} = 1$$

Unstable part  $f_a$  of  $f=t_{4i}^{-1}t_1$

$$f = \frac{-4/3(s+1)}{s-1} = -4/3 - \frac{8/3}{s-1}, \quad f_a = -\frac{8/3}{s-1}$$

Hence,  $X^0=0$ ,  $Q=0$  and

$$K(s) = \frac{y}{x} = \frac{-4/3(s+1)}{s+5/3} = -\frac{4(s+1)}{3s+5}$$

## REFERENCES

V.M. Adamjan, D.Z. Arov and M.G. Krein. Infinite block Hankel matrices and related extension problem. American Mathematical Society Translation, 111: 133-156, 1978.

Bala Subrahmanyam, *Finite Horizon  $H_\infty$  and Related Control Problems*, Birkhäuser, Boston, 1995.

Basar, T., and Bernhard, P.,  *$H_\infty$  Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, Birkhäuser, Boston, 1995.

Chen, B.M., *Robust and Optimal Control*, Springer-Verlag, 2000.

Colaneri, P., Geromel, J.C., and Locatelli, A., *Control Theory & Design: An  $H_2$  and  $H_\infty$  Viewpoint*, Academic Press, San Diego, 1997.

Dahleh, M.A., and Diaz-Bobillo, I., *Control of Uncertain Systems, A Linear Programming Approach*, Prentice Hall, Englewood Cliffs, 1995.

Davis, C., Kahan W.M., Weinberger, H.F., "Norm-preserving dilations and their applications to optimal error bounds", *SIAM Journal of Control and Optimization*, vol. 20, pp. 445-469, 1982.

Doyle J.C., "Analysis of feedback systems with structured uncertainties", *Proceedings of the IEEE, Part D*, vol. 129, pp. 242-250, 1982  
Hazony, D., "Zero cancellation synthesis using impedance operators", *IRE Trans. on Circuit Theory*, vol. 8, pp. 114-120, 1961.

Doyle J.C., *Lecture Notes in Advanced Multivariable Control*, Lecture Note ONR/Honeywell Workshop, Minneapolis, 1984.

Doyle, J.C., Francis, B.A., and Tannenbaum, A.R., *Feedback Control Theory*, MacMillan Publishing Co., New York, 1992.

J.C. Doyle, K. Glover, P.P. Khargonekar, B. Francis, "State-space solution to standard  $H_2$ - $H_\infty$  control problems", *IEEE Trans. Autom. Control*, AC-34, pp. 831-847, 1989.

D. Fragopoulos,  $H_\infty$  synthesis theory using polynomial system representations", PhD Thesis, University of Strathclyde, 1994.

Francis B.A., and Zames, G., "On  $L_2$ -optimal sensitivity theory for SISO feedback systems", *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 9-16, 1984.

B.A. Francis, "A course in  $H_\infty$  control theory", in *Lecture Notes in Control and Information Science*, Vol. 88, Springer-Verlag, Berlin, 1987.

Glover, K., "All optimal Hankel-norm approximations of linear multivariable systems and their  $L_2$ -error bounds, *International Journal of Control*, vol. 39, pp. 1115-1193, 1984.

M. Green, K. Glover, D.J.N. Limebeer, J.C. Doyle, "A J-spectral factorization approach to  $H_\infty$  control", *SIAM J. Contr. And Optimization*, 28, pp. 1350-1371, 1990.

M.J. Grimble and V. Kucera, "Polynomial methods for control systems design", Springer Verlag 1996.

Helton, J.W.; and Merino, O., *Classical Control Using  $H_2$  Methods: Theory, Optimization and Design*, SIAM, Philadelphia, 1988.

Heyde, R.A., *H<sub>∞</sub> Aerospace Control Design: A VSTOL Flight Application*, Springer-Verlag, New-York, 1995.

Kimura, H., Okinishi, F., "Chain scattering approach to control system design", In *Trends in Control, An European Perspective*, A.Isidori Ed., pp. 151-171, Springer Verlag,, Berlin, 1995.

Kimura, H., "Conjugation, interpolation and model-matching in H<sub>∞</sub>", *International Journal of Control*, vol. 49, pp. 269-307, 1989.

H. Kwakernaak, M. Sebek, "Polynomial J-spectral factorization", *IEEE Trans. Autom. Control*, AC-39, pp. 315-328, 1994.

McFarlane D., and Glover, K, "A loop shaping design procedure using H<sub>∞</sub> synthesis", *IEEE Trans. Automat. Control*, vol. 37, pp. 759-769, 1992.

G. Mensma, "Frequency-domain methods in H<sub>∞</sub> Control", PhD Thesis, University of Twente, 1993.

Mustafa, D. and Glover, K., *Minimum Entropy H<sub>∞</sub> Control*, Lecture Notes in Control and Information Sciences, Vol. 16, Springer-Verlag, 1990.

R. Nevanlinna, "Über beschränkte Funktionen, die in gegebenen Punkten vorgeschriebene Werte annehmen", *Ann. Acad. Sci. Fenn.*, Vol. 13, pp. 27-43, 1919.

G. Pick, "Über Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktions wert bewirkt sind", *Mathematische Ann.*, Vol. 77, pp. 7-23, 1916.

Stoorgvogel, A., *The  $H_{\infty}$  Control Problem: A State Space Approach*, Prentice Hall, Englewood Cliffs, New Jersey, 1992.

Van Keulen, B.,  *$H_{\infty}$  Control for Distributed Parameter Systems: A State Space Approach*, Birkhäuser, Boston, 1993.

W.M. Vidyasagar, "Control system synthesis: a factorization approach", MIT Press, Cambridge, 1985.

I. Yaesh, "Linear Optimal Control and Estimation in the Minimum  $H_{\infty}$  norm sense", Phd thesis, Tel Aviv University, 1992.

Zames, G., "Feedback and optimal sensitivity: Model reference transformations, multiplicity seminorms, and approximation inverses", *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 301-320, 1981.

# **H<sub>∞</sub> FILTERING IN DISCRETE-TIME**

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# SUMMARY

- **$H_\infty$  analysis**

*$H_\infty$  norm. Small gain theorem. Algebraic and difference analysis Riccati equation*

- **$H_\infty$  filtering (infinite horizon)**

*Stochastic filtering. Basic prediction problem.  $J$ -spectral factorization. Duality. Krein spaces. Risk sensitivity.*

- **Mixed  $H_2/H_\infty$  filtering**

*Convex optimization.  $\alpha$  procedure*

- **$H_\infty$  filtering (finite horizon)**

*Feasibility. Convergence. Gamma Switching*

- **Conclusions**



# H<sub>∞</sub> NORM

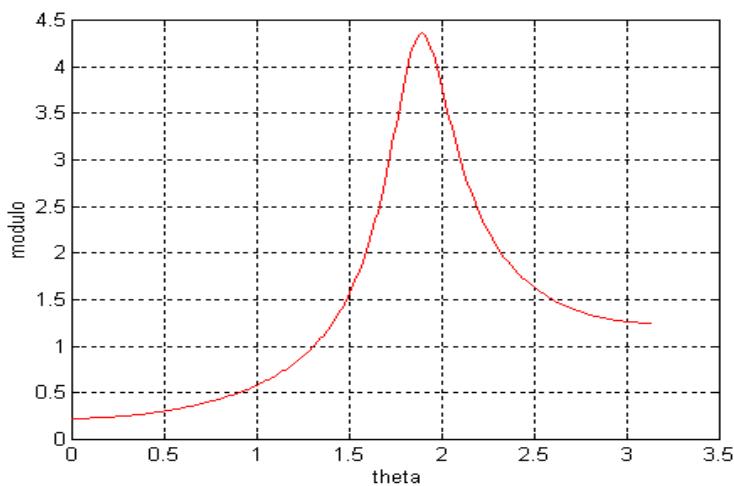
$$x(t+1) = Ax(t) + Bw(t)$$

$$z(t) = Cx(t) + Dw(t)$$

$$G(z) = C(zI - A)^{-1}B + D$$

$$\|G(z)\|_{\infty}^2 = \sup_{J \in [0, p]} \mathbf{I}_{\max}(G(e^{jJ})G(e^{-jJ})') = \sup_{w \in l_2} \frac{\|y_F\|_2^2}{\|w\|_2^2} = \sup_{w(z) \in l_2} \frac{\|G(z)w(z)\|_2^2}{\|w(z)\|_2^2}$$

$$G(z) = \frac{z - 0.5}{z^2 + 0.5z + 0.7}$$

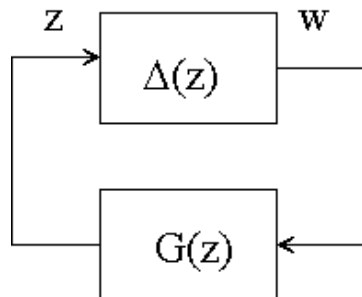


*If  $w(t)$  is a white noise, the norm represents the square-root of the peak value of the spectrum of the output*

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For the frequency-domain definition it is enough to require that  $G(z)$  don't have poles on the unit circle ( $L_{\infty}$ ). It is easy to reckonize that the  $H_{\infty}$  norm of a function in  $L_{\infty}$  coincides with the norm of its stable part  $G_0(z)$  in the inner-outer factorization  $G(z) = G_0(z)G_1(z)$ . The time-domain definition requires the external stability of the system. The symbol  $l_2$  indicates the set of the square-summable discrete-time functions in  $Z^+$ . Notice that if  $w$  belongs to  $l_2$  and the system is asymptotically stable, the forced  $y_F$  is in  $l_2$  as well.

# SMALL GAIN THEOREM: robust stability



Let  $G(z)$  and  $D(z)$  stable. Then the closed-loop system  $(D, G)$  is stable for each  $\|D(z)\|_{\infty} < \alpha$  iff  $\|G(z)\|_{\infty} < \alpha^{-1}$ .

$$\|G(z)\|_{\infty}^2 = \sup_{w \in l_2} \frac{\|y_F\|_2^2}{\|w\|_2^2} < \mathbf{g}^2 \quad \rightarrow \quad \|y_F\|_2^2 - \mathbf{g}^2 \|w\|_2^2 < 0, \forall w \in l_2$$

## Theorem 1

Let  $A$  be stable and  $\|G(z)\|_{\infty} < \mathbf{g}$ . Then,

$$\sup_{w \in l_2} \|y_F\|_2^2 - \mathbf{g}^2 \|w\|_2^2 = x_0' P x_0$$

where  $P \geq 0$  is the stabilizing solution of the Riccati equation

$$P = A' P A + (A' P B + C' D)(\mathbf{g}^2 I - D' D - B' P B)^{-1} (A' P B + C' D)' + C' C$$

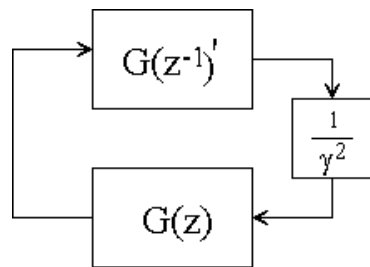
The worst disturbance is

$$w_{\text{worst}}(t) = (\mathbf{g}^2 I - D' D - B' P B)^{-1} (B' P A + D' C) x(t)$$

The fact that the  $H_{\infty}$  is bounded is equivalent to the existence of the stabilizing solution of the equation (see Theorem 2). The proof of the result directly follows from the equation by pre-multiplying it by  $x(t)'$  and post-multiplying it by  $x(t)$ , taking also into account the system equations and adding all quantities obtained from  $t=0$  to  $t=\infty$ .

# RICCATI EQUATION FOR THE $H_\infty$ ANALYSIS

*Descriptor symplectic system:*



## Theorem 2

*A is stable and  $\|G(z)\|_\infty < \mathbf{g}$  iff there exists  $Q \geq 0$  satisfying*

- (i) 
$$Q = AQA' + BB' + (AQC' + BD')(g^2 I - DD' - CQC')^{-1} (AQC' + BD)'$$
- (ii)  $g^2 I - DD' - CQC' > 0$  **feasibility**
- (iii)  $A + (AQC' + BD')(g^2 I - DD' - CQC')^{-1} C$  **stability**

## Notes:

The Riccati equation (or its dual) is associated with the symplectic system written above and described by equations below.

$$\begin{bmatrix} I & -C'(\mathbf{g}^2 I - DD')^{-1}C \\ 0 & A + BD'(\mathbf{g}^2 I - DD')^{-1}C \end{bmatrix} \mathbf{h}(t+1) = \begin{bmatrix} A'+C'(\mathbf{g}^2 I - DD')^{-1}DB' & 0 \\ -B(I - \mathbf{g}^{-2}D'D)B' & I \end{bmatrix} \mathbf{h}(t)$$

For the existence of the stabilizing solution of the Riccati equation it is necessary that such a system does not have characteristic values with unitary modulus.

From the equation it readily follows that:

$$\begin{aligned} \mathbf{g}^2 I - G(z)G(z^{-1})' &= \Psi(z) [\mathbf{g}^2 I - DD' - CQC'] \Psi(z^{-1})', \\ \Psi(z) &= I - C(zI - A)^{-1} (BD' + AQC') (\mathbf{g}^2 I - DD' - CQC')^{-1} \end{aligned}$$

This explains the fact that  $[\mathbf{g}^2 I - DD' - CQC']$  must be positive. As will be clear from the proof, the theorem can be formulated for the inequality

$$-Q + AQA' + (AQC' + BD') (\mathbf{g}^2 I - DD' - CQC')^{-1} (AQC' + BD')' \leq 0$$

without the stability condition (iii).

Notice that as  $\gamma \rightarrow \infty$  (the loop in the figure opens) the Riccati equation boils down to the Lyapunov equation associated with the  $H_2$  norm.

## Proof of Theorem 2

Assume that there exists a solution  $Q$  of the Riccati equation. Such an equation can be transformed in the following way (the computations are left to the reader):

$$\mathbf{g}^2 I - G(z)G(z^{-1})' = \Psi(z) [\mathbf{g}^2 I - DD' - CQC'] \Psi(z^{-1})',$$

where

$$\Psi(z) = I - C(zI - A)^{-1} (BD' + AQC') (\mathbf{g}^2 I - DD' - CQC')^{-1}.$$

Hence, on the basis of the property  $\mathbf{g}^2 I - DD' - CQC' > 0$  and invertibility of  $\Psi(e^{j\mathbf{J}})$  we have that

$$\mathbf{g}^2 I - G(e^{j\mathbf{J}})G(e^{-j\mathbf{J}})' = \Psi(e^{j\mathbf{J}}) [\mathbf{g}^2 I - DD' - CQC'] \Psi(e^{-j\mathbf{J}})' > 0, \quad \forall \mathbf{J},$$

i.e. the thesis.

Vice-versa assume that  $\|G(z)_\infty\| < \mathbf{g}$ . We prove that the symplectic system

$$\begin{bmatrix} I & -C'(\mathbf{g}^2 I - DD')^{-1}C \\ 0 & A + BD'(\mathbf{g}^2 I - DD')^{-1}C \end{bmatrix} \mathbf{h}(t+1) = \begin{bmatrix} A'+C'(\mathbf{g}^2 I - DD')^{-1}DB' & 0 \\ -B(I - \mathbf{g}^{-2}D'D)B' & I \end{bmatrix} \mathbf{h}(t)$$

does not have characteristic values with unitary modulus. Indeed, assume, by contradiction,  $\lambda$  is such, i.e.

$$\mathbf{I}x - \mathbf{I}C'(\mathbf{g}^2\mathbf{I} - DD')^{-1}Cy = (A' + C'(\mathbf{g}^2\mathbf{I} - DD')^{-1}DB')x$$

$$\mathbf{I}(A + BD'(\mathbf{g}^2\mathbf{I} - DD')^{-1}C)y = y - B(I - \mathbf{g}^{-2}D'D)B'x$$

for a certain  $[x \ y]' \neq 0$  and define  $p = (\mathbf{g}^2\mathbf{I} - DD')^{-1}(DB'x + \mathbf{I}Cy)$ . It results  $G(\mathbf{I}^J)G(\mathbf{I})' = \mathbf{g}^2\mathbf{I}$ , which is a contradiction unless  $p=0$ . On the other hand,  $p=0$  implies  $A'x = \mathbf{I}x$ , and, since  $A$  is stable,  $x=0$ . Finally,  $p=0$  and  $x=0$  imply  $(\mathbf{I}A - \mathbf{I})y = 0$  so that  $y=0$  ( $A$  is stable). Hence,  $x=0$  and  $y=0$  is a contradiction.

Since the symplectic system does not have characteristic values on the unit circle, there are  $n$  characteristic values inside (possibly in zero) and  $n$  outside (the reciprocals) the unit circle (possibly at infinity). Grouping the  $2n$ -dimensional vectors associated with the  $n$  characteristic values inside the unit circle, one can construct a matrix  $[X \ Y]'$ , whose range is a  $2n$ -dimensional subspace. It follows

$$\begin{bmatrix} \mathbf{I} & -C'(\mathbf{g}^2\mathbf{I} - DD')^{-1}C \\ 0 & A + BD'(\mathbf{g}^2\mathbf{I} - DD')^{-1}C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} T = \begin{bmatrix} A' + C'(\mathbf{g}^2\mathbf{I} - DD')^{-1}DB' & 0 \\ -B(I - \mathbf{g}^{-2}D'D)B' & \mathbf{I} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

where  $T$  is a matrix with eigenvalues inside the unit circle (stable matrix). Consider now the solutions of the system  $r(t+1) = Tr(t)$ , i.e.  $r(t) = T^t r(0)$  and left multiply by  $r(t)$  the two matrix equations above. Next, define  $p(t) = B'Xr(t)$  and  $q(t) = CYr(t)$ . It follows

$$p(t+1) = A'p(t) + C'v_1(t), \quad v_2(t) = B'p(t) + D'v_1(t)$$

$$Aq(t+1) = q(t) - Bv_2(t), \quad v_1(t) = \frac{1}{\mathbf{g}^2}(Cq(t+1) + Dv_2(t))$$

from which, passing to the Z-transform

$$G(z)'G(z^{-1})v_1(z) = \mathbf{g}^2v_1(z), \quad G(z^{-1})G(z)'v_2(z) = \mathbf{g}^2v_2(z).$$

From the assumption it is  $v_1(t) = 0, v_2(t) = 0$ , so that  $B'X = 0$  and  $C'Y = 0$ . Hence, from the symplectic system it follows  $A'X = XT, AY = Y$ . Assume that there exists a vector  $d$  such that  $Xd = 0$  and take the minimal degree polynomial  $\mathbf{v}(T)$  of  $T$  such that  $\mathbf{v}(T)d = 0$ . Such a polynomial can be written as  $\mathbf{v}(T) = (\mathbf{I} - T)\mathbf{m}(T)$ . For the minimality of  $\mathbf{v}(T)$  it results  $\mathbf{m}(T)d \neq 0$ . Then,  $AYTd = \mathbf{I}AYd = Yd$ . Since  $\lambda$  has modulus less than 1,  $\mathbf{I}^J$  has modulus greater than one so that this equation contradicts the stability of  $A'$ , unless  $Yd = 0$ . On the other hand, this last conclusion, together with  $Xd = 0$  contradicts the  $n$ -dimensionality of the range of  $[X \ Y]'$ .

# THE DIFFERENCE EQUATION

$$x(t+1) = Ax(t) + Bw(t)$$

$$y(t) = Cx(t) + Dw(t)$$

$$Q(t+1) = AQ(t)A' + (AQ(t)C' + BD')(\mathbf{g}^2 I - DD' - CQ(t)C')^{-1}(AQ(t)C' + BD')' + BB'$$

$$Q(0) = Q_0 > 0$$

### Theorem 3

$$J = \sum_{t=0}^N y(t)' y(t) - \mathbf{g}^2 \sum_{t=0}^N w(t)' w(t) - \mathbf{g}^2 x(0)' Q_0^{-1} x(0) < 0, \quad \forall (w(\cdot), x(0)) \neq 0$$

iff there exists  $Q(t) \geq 0$  in  $[0, N]$  such that  $\mathbf{g}^2 I - DD' - CQ(t)C' > 0$

### Proof of Theorem 3

Consider the cost.

$$\tilde{J}(N) = \sum_{t=0}^N z(t)' z(t) - \mathbf{g}^2 \sum_{t=0}^N w(t)' w(t) + x(N+1)' P_{N+1} x(N+1) < 0, x(0) = 0, \forall w(\cdot) \neq 0$$

We proof the dual result:  $\tilde{J}(N) < 0, x(0) = 0, \forall w(\cdot) \neq 0$  iff there exists the solution of the equation

$$\begin{aligned} P(t) &= A' P(t+1) A + \\ &+ (A' P(t+1) B + C' D) (\mathbf{g}^2 I - D' D - B' P(t+1) B)^{-1} (A' P(t+1) B + C' D)' + C' C, \\ P(N+1) &= P_{N+1} > 0 \end{aligned}$$

with  $\gamma^2 I - D' D - B' P(t+1) B > 0$ , in  $[0, N+1]$ . Sufficient condition: if there exists the solution with the indicated properties, then pre-premultiplying the equation by  $x(t)'$  and post-multiplying by  $x(t)$ , and summing all terms, it results

$$\sum_{t=0}^N z(t)' z(t) - \mathbf{g}^2 \sum_{t=0}^N w(t)' w(t) + x(N+1)' P_{N+1} x(N+1) = x(0)' P(0) x(0) - \sum_{t=0}^N (w(t) - \tilde{w}(t))' (w(t) - \tilde{w}(t))$$

(\*)

where

$$\tilde{w}(t) = (\mathbf{g}^2 I - D' D - B' P(t+1) B)^{-1} (A' P(t+1) B + C' D)' x(t).$$

Hence, being  $x(0)=0$  we have  $\tilde{J}(N) < 0, \forall w(\cdot) \neq 0$ .

Vice-versa, assume that  $\tilde{J}(N) < 0, \forall w(\cdot) \neq 0$  and take  $w(\cdot)=0$  in  $[0, N-1]$ . Being  $x(0)=0$ , with such a choice we have

$$0 > w(N)' \left[ -\gamma^2 I + D' D + B P_{N+1} B' \right] w(N), \forall w(N) \neq 0$$

so that

$$-\gamma^2 I + D' D + B P_{N+1} B' > 0.$$

Furthermore, from

$$\tilde{J}(N) = \tilde{J}(N-1) - (w(N) - \tilde{w}(N))' (w(N) - \tilde{w}(N)) < 0, \forall w(\cdot) \in [0, N]$$

it turns out that

$$\tilde{J}(N-1) > 0, \forall w(\cdot) \in [0, N-1].$$

By iterating this reasoning we have

$$\gamma^2 I - D' D - B' P(t+1) B > 0$$

in  $[0, N]$ . Theorem 3 is proven from (\*) by noticing that

$$P(t) = Q(t)^{-1} \gamma^2.$$

# STOCHASTIC FILTERING

$$\begin{cases} x(t+1) = Ax(t) + v_1(t) \\ z(t) = Lx(t) \\ \bar{y}(t) = \bar{C}x(t) + v_2(t) \end{cases}$$

where  $v_1(t), v_2(t)$  are zero-mean white gaussian noises . Moreover,

$$E \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} V_1 & V_{12} \\ V_{12}' & V_2 \end{bmatrix}, \quad V_2 > 0$$

The problem is to find an estimate  $\hat{z}(t)$  in  $H_\infty$  of a linear combination  $z(t) = Lx(t)$  of the state, i.e. one wants to find an estimator based on the observations of  $\{\bar{y}(i), i \leq t - N\}$  in such a way that the maximum of the output spectrum of the error  $e = z - \hat{z}$  due to  $[v_1' \ v_2']'$  is minimized (or bounded from above by a certain positive scalar  $\gamma$ ).

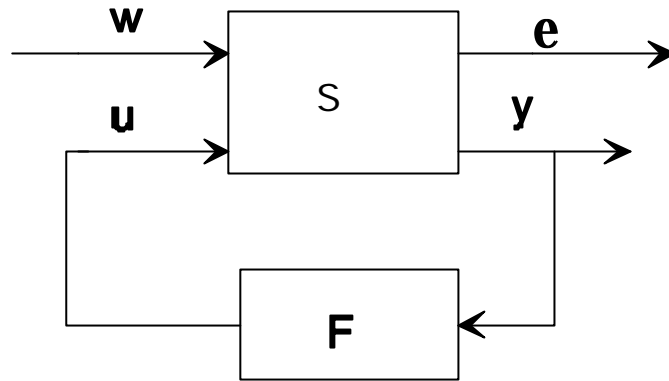
Letting

$$\begin{aligned} u(t) &= -\hat{z}(t), \quad e(t) = Lx(t) + u(t), \quad y(t) = V_2^{-1/2} \bar{y}(t) \\ D &= \begin{bmatrix} 0 & V_2^{-1/2} \end{bmatrix}, \quad C = \bar{C} \\ B &= \begin{bmatrix} (V_1 - V_{12} V_2^{-1} V_{12}')^{1/2} & V_{12} V_2^{-1/2} \end{bmatrix} \end{aligned}$$

and defining with  $w(t)$  a zero mean white gaussian noise with identity covariance, it follows:



# BASIC FILTERING PROBLEM



$$S : \begin{cases} x(t+1) = Ax(t) + Bw(t) \\ \mathbf{e}(t) = Lx(t) + u(t) \\ y(t) = Cx(t) + Dw(t) \end{cases}$$

$$F : \begin{cases} \hat{x}(t+1) = A_F \hat{x}(t) + B_F y(t-N) \\ u(t) = C_F \hat{x}(t) + D_F y(t-N) \end{cases}$$

*Basic problem: Find  $A_F, B_F, C_F, D_F$  such that*

- *$(S, F)$  is asymptotically stable*
- $\|T_{ew}\|_{\infty} < \mathbf{g}$

---

If  $z(t) = Lx(t)$  is the variable to be estimated and  $\hat{z}(t)$  the estimated variable, it follows  $u(t) = -\hat{z}(t)$ . Hence  $\mathbf{e}(t)$  represents the filtering error  $z(t) - \hat{z}(t) = \mathbf{e}(t)$ . If  $w(t)$  is a white noise, the problem is that of minimizing the peak value of the error spectrum. If  $w(t)$  is a generic signal in  $l_2$ , the problem is that of minimizing the  $l_2$  "gain" between the error and the disturbance.

# SOLUTION OF THE FILTERING PROBLEM

**Ipotesi:**  $DD' > 0$   
 $A = \text{stable}$   
 $(A, B, C, D)$  does not have unstable invariant zeros

## Theorem 4

There exists a filter  $\mathbf{F}$  solving the basic problem iff  $\exists Q \geq 0$  solution of

$$i) Q = AQA' + BB' - \begin{bmatrix} CQA' + DB' \\ LQA' \end{bmatrix} \begin{bmatrix} DD' + CQC' & CQL' \\ LQC' & -g^2 I + LQL' \end{bmatrix}^{-1} \begin{bmatrix} CQA' + DB' \\ LQA' \end{bmatrix}$$

$$ii) V = g^2 I - LQL' + LQC'(DD' + CQC')^{-1}CQL' > 0 \quad \text{filter feasibility}$$

$$iii) \hat{A} = A - \begin{bmatrix} CQA' + DB' \\ LQA' \end{bmatrix} \begin{bmatrix} DD' + CQC' & CQL' \\ LQC' & -g^2 I + LQL' \end{bmatrix}^{-1} \begin{bmatrix} C \\ L \end{bmatrix} \quad \text{stable}$$

## CENTRAL FILTER

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + F_1(y(t) - C\mathbf{x}(t)), \quad F_1 = (AQC' + BD')(CQC' + DD')^{-1}$$

$$\hat{\mathbf{z}}(t) = L\mathbf{x}(t) + F_2(y(t) - C\mathbf{x}(t)), \quad F_2 = LQC'(CQC' + DD')^{-1}$$

---

The assumption on the zeros is equivalent to the stabilizability of the pair

$(\tilde{A}, \tilde{B}), \tilde{A} = A - BD'(DD')^{-1}C, \tilde{B} = B(I - D'(DD')^{-1}D)$ . The assumption of stability of A can be weakened to the assumption of detectability of (A,C) if one is only interested in the stability of the filter.

## NOTES

- If  $G \neq 0$ , one obtains the central filter

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + F_1(y(t) - C\mathbf{x}(t)) + Gu(t)$$

$$\hat{\mathbf{z}}(t) = L\mathbf{x}(t) + F_2(y(t) - C\mathbf{x}(t))$$

- As  $g \rightarrow \infty$  the  $H_2$  filtering is obtained, where  $F_1$  e  $F_2$  depend on the stabilizing solution of the standard Riccati equation:

$$Q = AQA' + BB' - (CQA' + DB')'(DD' + CQC)^{-1}(CQA' + DB')$$

- Notice that in the  $H_{\mathbf{y}}$  filter the solution depends on  $L$ , i.e. on the particular linear combination of the state to be estimated. This is a difference between  $H_2$ , and  $H_{\mathbf{y}}$  design.

- The “gains”  $F_1$  and  $F_2$  are related by:

$$F_2 = LK$$

$$F_1 = [A - BD'(DD')^{-1}C]K - BD'(DD')^{-1}$$

with

$$K = QC'(CQC' + DD')^{-1}$$

In the uncorrelated case ( $DB'=0$ ) these relations become

$$F_2 = LK, \quad F_1 = AK$$

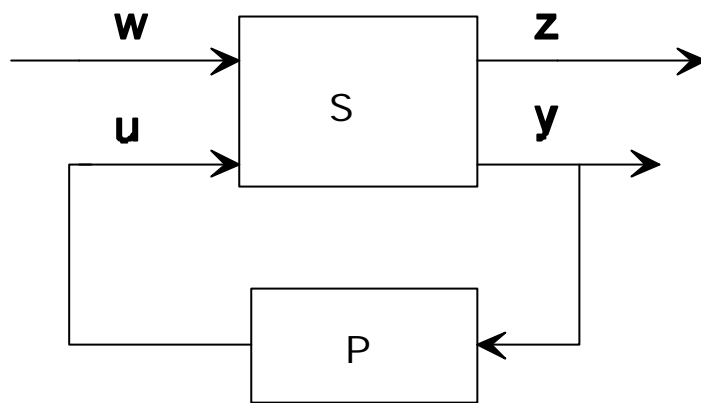
Hence

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + AK[y(t) - C\mathbf{x}(t)]$$

$$\hat{\mathbf{r}}(t) = L\mathbf{x}(t) + LK[y(t) - C\mathbf{x}(t)]$$

# BASIC ONE-STEP PREDICTION PROBLEM

Assume that we want to estimate  $z(t) = Lx(t)$  on the basis of the data  $y(i), i \in t$ .



$$S : \begin{cases} x(t+1) = Ax(t) + Bw(t) + Gu(t) \\ \mathbf{e}(t) = Lx(t) + u(t) \\ y(t) = Cx(t) + Dw(t) \end{cases}$$

$$P : \begin{cases} \tilde{x}(t+1) = A_p \tilde{x}(t) + B_p y(t) \\ u(t) = C_p \tilde{x}(t) \end{cases}$$

*Basic problem: Find  $A_p, B_p, C_p$  such that:*

- $(S, F)$  asymptotically stable
- $\|T_{rw}\|_{\infty} < \gamma$

---

If  $z(t) = Lx(t)$  is the variable to be estimated and  $\tilde{z}(t)$  is the estimated variable, then  $u(t) = -\tilde{z}(t)$  so that  $\mathbf{e}(t)$  represents the prediction error. .

# SOLUTION OF THE ONE-STEP PREDICTION PROBLEM

## Assumption:

$A = \text{stable}, DD' > 0, (A, B, C, D)$  without unstable zeros

## Theorem 5

There exists a one-step predictor  $\mathbf{P}$  solving the problem iff  $\exists Q^30$  satisfying

$$i) Q = AQA' + BB' - \begin{bmatrix} CQA' + DB' \\ LQA' \end{bmatrix} \begin{bmatrix} DD' + CQC' & CQL' \\ LQC' & -g^2 I + LQL' \end{bmatrix}^{-1} \begin{bmatrix} CQA' + DB' \\ LQA' \end{bmatrix}$$

$$ii) g^2 I - LQL' > 0$$

predictor feasibility

$$iii) \hat{A} = A - \begin{bmatrix} CQA' + DB' \\ LQA' \end{bmatrix} \begin{bmatrix} DD' + CQC' & CQL' \\ LQC' & -g^2 I + LQL' \end{bmatrix}^{-1} \begin{bmatrix} C \\ -L \end{bmatrix} \text{ stable}$$

## CENTRAL PREDICTOR

$$\mathbf{h}(t+1) = A\mathbf{h}(t) + F_3(y(t) - C\mathbf{h}(t))$$

$$\tilde{\mathbf{z}}(t) = L\mathbf{h}(t)$$

$$F_3 = (AQC' + BD')(CQC' + DD')^{-1} + A_X QL' V^{-1} LQC' (CQC' + DD')^{-1}$$

$$A_X = A - (AQC' + BD')(CQC' + DD')^{-1} C$$

$$V = g^2 I - LQL' + LQC' (DD' + CQC')^{-1} CQL'$$

---

The assumption on the zeros is equivalent to the stabilizability of the pair  $(\tilde{A}, \tilde{B})$ ,  $\tilde{A} = A - BD'(DD')^{-1}C$ ,  $\tilde{B} = B(I - D'(DD')^{-1}D)$ . The assumption of stability of  $A$  can be weakened to that of detectability of  $(A, C)$ , if one is only interested to the stability of the predictor.

# NOTES

- If  $G \neq 0$ , the so-called central predictor is obtained

$$\begin{aligned} \mathbf{m}(t+1) &= A\mathbf{h}(t) + F_3 (y(t) - C\mathbf{h}(t)) + G\mathbf{u}(t) \\ \tilde{\mathbf{z}}(t) &= L\mathbf{h}(t) \end{aligned}$$

- When  $g \rightarrow \infty$  the  $H_2$  predictor is recovered. Moreover,  $F_3 \rightarrow F_1$ , i.e. the gain of the predictor and the gain of the filter coincide and depend on the stabilizing solution of the standard Riccati equation:

$$Q = AQA' + BB' - (CQA' + DB')'(DD' + CQC)^{-1}(CQA' + DB')$$

- Notice that the  $H_\infty$  predictor depends on  $L$ , i.e. on the particular linear combination of the state to be estimated. This is a difference between  $H_2$  and  $H_\infty$  design.
- The “gain”  $F_3$  of the predictor and the “gains”  $F_1$  and  $F_2$  of the filter are related by:

$$F_3 = F_1 + [A - F_1C]QL'V^{-1}F_2$$

---

In the uncorrelated case ( $DB'=0$ ) and under the assumption of reachability of  $(A,B)$ , the Riccati equation of Theorem 3 can be rewritten in the following way:

$$Q = A \left[ Q^{-1} + C'(DD')^{-1}C - \frac{1}{g^2}L'L \right]^{-1} A' + BB'$$

**Feasibility condition for the predictor**  $g^2I - LQL' > 0$

**Feasibility condition for the filter**  $Q^{-1} + C'C - L'Lg^{-2} > 0$

**Filter/Predictor**

$$F_1 = A(Q^{-1} + C'C)^{-1}C', \quad F_2 = L(Q^{-1} + C'C)^{-1}C', \quad F_3 = A(Q^{-1} + C'C - \frac{1}{g^2}L'L)C'$$

## PREDICTOR VS FILTER

The Riccati equation is the same, but the feasibility conditions are different

### System for the prediction error

$$\mathbf{m}(t+1) = (A - F_3 C)\mathbf{m}(t) + (F_3 C - B)w(t)$$

$$e(t) = L\mathbf{m}(t)$$

*feasibility (analysis) :*  $\mathbf{g}^2 I - LQL' > 0$

### System for the filtering error

$$\mathbf{h}(t+1) = (A - F_1 C)\mathbf{h}(t) + (F_1 D - B)w(t)$$

$$e(t) = (L - F_2 C)\mathbf{h}(t) + F_2 Dw(t)$$

*feasibility (analysis) :*

$$\mathbf{g}^2 I - LQL' + LQC'(DD' + CQC')^{-1}CQL' - (F_2 - \bar{F}_2)(DD' + CQC')(F_2 - \bar{F}_2)' > 0$$

$$\bar{F}_2 = LQC'(CQC' + DD')^{-1}$$

- 1) At  $\gamma$  fixed, the feasibility condition of the predictor is more stringent than that of the filter. They coincide as  $\gamma$  tends to infinity, to witness the fact that the Kalman predictor and filter share the same existence conditions.
- 2) Assuming the existence of the stabilizing solution of the Riccati equation satisfying the feasibility condition, the sufficient part of Theorem 3 is proved from the analysis result. Indeed, the analysis Riccati equation coincides with the synthesis Riccati equation once the gains  $F_1, F_2$  (for the filter) or  $F_3$  (for the predictor) are substituted there.

**Proof of Theorem 4/5**

For simplicity we study the case where  $DB'=0$ ,  $DD'=I$ ,  $(A,B)$  reachable. Moreover, we limit the proof to filters (predictors) in observer form. This is a restriction, but the analysis of all filters (predictors) ensuring the norm bounded is far from being trivial and is not within our scope. Notice that if  $F_1=AK$  e  $F_2=LK$  are the characteristic gains of a generic filter, the transfer function from the disturbance to the error is

$$T_{\varepsilon w}=(L-LKC)(zI-A+AKC)^{-1}(AKD-B)+LKD$$

So that the analysis Riccati equation is

$$Q = A( X^{-1} - L' L \mathbf{g}^{-2} )^{-1} A' + BB', \quad X = ( I - KC ) Q ( I - KC )' + KK' \quad (1)$$

with the feasibility condition

$$\mathbf{g}^2 I - LXL' > 0 \quad (2)$$

and asymptotic stability of

$$A( I - KC ) + AXL' ( \mathbf{g}^2 I - LXL' )^{-1} ( I - KC )' L' \quad (3)$$

On the other hand, if  $F_3$  is the gain of a generic one-step predictor, the transfer function is

$$T_{\varepsilon w}=L(zI-A+F_3C)^{-1}(F_3D-B),$$

and then the analysis Riccati equation is:

$$Q = A(Q^{-1} + C'C - L'L\mathbf{g}^{-2})^{-1} + BB' + (F_3 - AYC'(I + CYC')^{-1})(I + CYC')(F_3 - AYC'(I + CYC')^{-1})'$$

$$Y = (Q^{-1} - L'L\mathbf{g}^{-2}) \quad (4)$$

with the feasibility condition

$$\mathbf{g}^2 I - LQL' > 0 \quad (5)$$

and asymptotic stability of

$$A - F_3C + (A + F_3C)QL' ( \mathbf{g}^2 I - LQL' )^{-1} L' \quad (6)$$

Now, assume that there exists  $Q \geq 0$  satisfying  $Q=A[Q^{-1}+C'C-L'L\gamma^2]^{-1}+BB'$ , and such that  $Q^{-1}+C'C-L'L\gamma^2 > 0$  and  $A(Q^{-1}+C'C-L'L\gamma^2)^{-1} Q^{-1}$  stable. Take the filter characterized by  $F_1=AK$ ,  $F_2=LK$ , with  $K=(Q^{-1}+C'C)^{-1}C'$ . From (1) it results  $X=(Q^{-1}+C'C)^{-1}$  and hence

$$Q = AXA' + BB' + AXL' ( \mathbf{g}^2 I - LXL' )^{-1} LXA' = A( X^{-1} - L' L \mathbf{g}^{-2} )^{-1} A' + BB' = A( Q^{-1} + C' C - L' L \mathbf{g}^{-2} )^{-1} A' + BB'$$

Therefore, the analysis Riccati equation admits a positive semidefinite solution with  $\mathbf{g}^2 I - LXL' = \mathbf{g}^2 I - LQL' LQC' ( I + CQC' )^{-1} CQL' > 0$ ,

so that the feasibility condition is satisfied. Also the stability condition (3) is satisfied.

Hence the norm is less than  $\gamma$ .

Assume now that there exists  $Q \geq 0$  satisfying  $Q=A[Q^{-1}+C'C-L'L\gamma^2]^{-1}+BB'$ , and such that  $Q^{-1}-L'L\gamma^2 > 0$  and  $A(Q^{-1}+C'C-L'L\gamma^2)^{-1} Q^{-1}$  stable. Take a predictor defined by  $F_3=A(Q^{-1}+C'C-L'L\gamma^2)^{-1}C'$ . From (4) it results that  $F_3=AYC'(I+CYC')^{-1}$  and hence  $Q=AYA'+BB'-AYC'(I+CYC')^{-1}CYA'=A(Q^{-1}+C'C-L'L\gamma^2)^{-1}A'+BB'$ . The analysis Riccati equation admits a positive semidefinite solution and  $Q^{-1}-L'L\gamma^2 > 0$  coincides with the feasibility condition (5) for the predictor. Also the stability condition (6) is satisfied.

Hence the norm is less than  $\gamma$ .

Vice-versa, assume that there exists a filter such that the norm is less than  $\gamma$ . Then, there exists  $Q$  solving (1)-(3). On the other hand,  $X=(Q^{-1}+C'C)^{-1}+(K-(Q^{-1}+C'C)^{-1}C')(I+CQC')(K-(Q^{-1}+C'C)^{-1}C') \geq (Q^{-1}+C'C)^{-1}$  and hence  $X^{-1} \leq Q^{-1}+C'C$  implies that  $Q$  solves the inequality  $Q \geq A(Q^{-1}+C'C-L'L\gamma^2)^{-1}A'+BB'$  whereas  $\gamma^2 I - LXL' > 0$  with  $X > 0$  implies that  $Q^{-1}+C'C-L'L\gamma^2 > 0$ . The existence of  $Q$  solving  $Q=A(Q^{-1}+C'C-L'L\gamma^2)^{-1}A'+BB'$  with  $Q^{-1}+C'C-L'L\gamma^2 > 0$  and satisfying the stability condition follows from standard monotonicity results of the solutions of the Riccati equation.

Finally, assume that there exists a predictor such that the norm is less than  $\gamma$ . Then, there exists  $Q$  solving (4)-(6). Hence  $Q \geq A(Q^{-1}A'+C'C-L'L\gamma^2)^{-1}A'+BB'$  with  $\gamma^2 I - LQL' > 0$ . The existence of  $Q$  solving  $Q=A(Q^{-1}A'+C'C-L'L\gamma^2)^{-1}A'+BB'$  with  $\gamma^2 I - LQL' > 0$  and satisfying the stability condition follows from standard monotonicity results of the solutions of the Riccati equation.



## J-SPECTRAL FACTORIZATION

The Riccati equation underlying the problem solution can be compactly written as:

$$Q = AQA' + BB' - [AQ\tilde{C}' + B\tilde{D}'] [R + \tilde{C}Q\tilde{C}']^{-1} [\tilde{C}QA' + \tilde{D}B']$$

$$\tilde{C} = \begin{bmatrix} C \\ L \end{bmatrix}, \tilde{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}, R = \begin{bmatrix} DD' & 0 \\ 0 & -\mathbf{g}^2 I \end{bmatrix}$$

Let define:

$$\begin{aligned} H_1(z) &= D + C(zI - A)^{-1} B \\ H_2(z) &= L(zI - A)^{-1} B \end{aligned} \quad H(z) = \begin{bmatrix} H_1(z) & 0 \\ H_2(z) & I \end{bmatrix} \quad J = \begin{bmatrix} I & 0 \\ 0 & -\mathbf{g}^2 I \end{bmatrix}$$

### Theorem 6

Assume that  $A$  is stable. There exists a stable filter  $F(z)$  such that  $H_2(z) - F(z)H_1(z)$  is stable and

$$\begin{aligned} H_2(z) - F(z)H_1(z) & \text{ stable} \\ \|H_2(z) - F(z)H_1(z)\|_{\infty} & < \mathbf{g} \end{aligned}$$

if and only if there exists a square  $\Omega(z)$ , stable with stable inverse, with stable  $[\Omega(z)^{-1}]_{11}$ , solving the factorization problem

$$H(z)JH(z^{-1})' = \Omega(z)J\Omega(z^{-1})'$$

All feasible filters can be parametrized in the following way:

$$F(z) = (\Omega_{21}(z) - \Omega_{22}(z)U(z))(\Omega_{11}(z) - \Omega_{12}(z)U(z))^{-1}, U(z) \in H_{\infty}, \|U(z)\|_{\infty} < \mathbf{g}$$

## Notes:

From  $z = H_2(z)w$ ,  $y = H_1(z)w$ ,  $\hat{z} = F(z)y$ , it results  $z - \hat{z} = (H_1(z) - F(z)H_2(z))w$  and then the filtering problem can be equivalently rewritten as a model matching problem.

### Proof of Theorem 6

Assume that there exists a feasible filter. We limit ourselves to the case where the filter is in the observer form. From Theorem 4 we know that there exists a positive semidefinite solution  $Q$  of the Riccati equation. Moreover, such a solution is stabilizing and feasible. It is a matter of cumbersome computations to show that

$$\Omega(z) = \left[ I + \tilde{C}(zI - A)^{-1}(B\tilde{D}' + AQC\tilde{C}') \right] (R + \tilde{C}Q\tilde{C}')^{-1} \begin{bmatrix} (DD' + CQC')^{1/2} & 0 \\ LQC'(DD' + CQC')^{-1/2} & \frac{1}{\mathbf{g}}V^{1/2} \end{bmatrix}$$

$$V = \mathbf{g}^2 I - LQL' + LQC'(DD' + CQC')^{-1}CQL'$$

is a J-spectral factor. The stability of  $\Omega(z)$  follows from stability of  $A$ . Stability of  $\Omega(z)^{-1}$  and  $[\Omega(z)^{-1}]_{11}$  follows from  $Q$  being a stabilizing solution. Existence of  $V^{1/2}$  follows from the feasibility condition.

Vice-versa, assume that  $\Omega(z)$  with the given properties exists and take the formula defining all filters  $F(z)$ . It follows

$$[F(z) \quad -I] = -(\Omega_{22}(z) - \Omega_{21}(z)\Omega_{11}(z)^{-1}\Omega_{12}(z)) \left( I - U(z)\Omega_{11}(z)^{-1}\Omega_{21}(z) \right)^{-1} [U(z) \quad I] \Omega(z)^{-1} \quad (*)$$

Notice that, thanks to the small gain theorem,

$$\left( I - U(z)\Omega_{11}(z)^{-1}\Omega_{21}(z) \right)^{-1}$$

is stable. Indeed,  $U(z)$  is stable and  $\|U(z)\|_\infty < \gamma$  and  $\|\Omega_{11}(z)^{-1}\Omega_{21}(z)\| < \gamma^{-1}$  since  $\Omega_{11}(z)\Omega_{11}(z)^{-1} - \gamma^2\Omega_{11}(z)\Omega_{11}(z)^{-1} = H_1(z)H_1(z)^{-1} > 0$  on the unit circle. Hence, stability of  $F(z)$  follows from (\*).

$$F(z)H_1(z) - H_2(z) = [F(z) \quad -I]H(z) = (\Omega_{22}(z) - \Omega_{21}(z)\Omega_{11}(z)^{-1}\Omega_{12}(z)) \left( I - U(z)\Omega_{11}(z)^{-1}\Omega_{21}(z) \right)^{-1} [U(z) \quad I] \Omega(z)^{-1} H(z)$$

so that  $F(z)H_1(z) - H_2(z)$  is stable. Finally,

$$(F(z)H_1(z) - H_2(z))J(F(z)H_1(z) - H_2(z))^\sim = [F(z) \quad -I]H(z)JH(z)^\sim \begin{bmatrix} F(z)^\sim \\ -I \end{bmatrix} = [F(z) \quad -I] \Omega(z) J \Omega(z)^\sim \begin{bmatrix} F(z)^\sim \\ -I \end{bmatrix} = (F(z)\Omega_{12}(z) - \Omega_{22}(z)) [U(z)U(z)^\sim - \mathbf{g}^2 I] (F(z)\Omega_{12}(z) - \Omega_{22}(z))^\sim$$

so that the norm of  $F(z)H_1(z) - H_2(z)$  is less than  $\gamma$ .

# DUALITY'

*The prediction problem is dual with respect to the full-state control problem ( $x$  is measurable).*

*The filtering problem is dual with respect to the full-information control problem ( $x$  and  $w$  are measurable).*

$$S : \begin{cases} x(t+1) = Ax(t) + Bw(t) + \\ \mathbf{e}(t) = Lx(t) + u(t) \\ y(t) = Cx(t) + Dw(t) \end{cases} \quad \text{OE}$$

$$S'_R : \begin{cases} \tilde{x}(t+1) = A'\tilde{x}(t) + L'\tilde{\mathbf{e}}(t) + C'\tilde{y}(t) \\ \tilde{w}(t) = B'\tilde{x}(t) + D'\tilde{y}(t) \end{cases} \quad \text{FI-SF}$$

## Theorem 7

$\tilde{y}(t) = -K_1\tilde{x}(t) - K_2\tilde{\mathbf{e}}(t)$  stabilizes  $S'_R$  and  $\|T_{\tilde{w}\tilde{\mathbf{e}}}\|_\infty < \mathbf{g}$  iff

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + K_1'(y(t) - C\mathbf{x}(t))$$

$$-u(t) = L\mathbf{x}(t) + K_2'(y(t) - C\mathbf{x}(t))$$

stabilizes  $S$  and  $\|T_{\mathbf{e}v}\|_\infty < \mathbf{g}$ .

---

Notice that if  $K_2$  is zero (state-feedback), the filter is indeed a predictor. The proof of Theorem 7 is immediate. It is enough to verify that for any  $K_1$  and  $K_2$  it results  $T_{\mathbf{e}w} = T_{\tilde{w}\tilde{\mathbf{e}}}$ . Actually, an extended version of this result allows to link any full-information controller  $K(z)$  for system  $S'_R$  to a filter  $F(z)$  for system  $S$ .

# KREIN SPACES

*The filtering (prediction) problem can be interpreted as a Kalman filtering (prediction) problem in the Krein space (instead of Hilbert space).*

$$\begin{cases} x(t+1) = Ax(t) + v_1(t) \\ z(t) = Lx(t) \\ y(t) = \tilde{C}x(t) + v_2(t) \end{cases}$$

*with*

$$\tilde{C} = \begin{bmatrix} C \\ L \end{bmatrix}, \quad E \left[ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} v_1' & v_2' \end{bmatrix} \right] = \begin{bmatrix} BB' & BD' & 0 \\ DB' & DD' & 0 \\ 0 & 0 & -\mathbf{g}^2 I \end{bmatrix}$$

# RISK SENSITIVITY

$$x(t+1) = Ax(t) + v_1(t)$$

$$y(t) = Cx(t) + v_2(t)$$

where the noises are white gaussian noises with zero mean.

The problem of the risk sensitive filter is to minimize

$$\min_{\hat{z}(y)} \mathbf{m}(\mathbf{J}) = \min_{\hat{z}(y)} \left[ -\frac{2}{\mathbf{J}} \log \left( E \left( e^{-\frac{\mathbf{J}}{2} \|\hat{z}(y) - Lx\|^2} \right) \right) \right]$$

The parameter  $\mathbf{q}$  is the risk parameter

$\mathbf{q}=0$ risk-neutral	Kalman
$\mathbf{q}>0$ risk-seeking	modified Kalman
$\mathbf{q}<0$ risk-adverse	$H_\infty$

The solution of the problem with  $\mathbf{q}<0$  coincides with the solution of the  $H_\infty$  filter with  $\mathbf{g}^2 = -\mathbf{q}^{-1}$ .

## MIXED $H_2/H_\infty$ FILTERING

It makes sense to distinguish between two classes of disturbances: those whose statistics is known and those which are only requested to be bounded. For instance, in general, the spectral characteristics of the transducer errors are known whereas those related to modelling errors are unknown. Consequently, we can write a mixed filtering problem associated to a variable  $z(t)$ :

$$x(t+1) = Ax(t) + Bw(t) = Ax(t) + B_1w_1(t) + B_2w_2(t)$$

$$z(t) = Lx(t)$$

$$y(t) = Cx(t) + Dw(t) = Cx(t) + D_1w_1(t) + D_2w_2(t)$$

$$\mathbf{e}(t) = z(t) - \hat{z}(t)$$

### Mixed Problem

*Find a filter solving:*

$$\inf \left\{ \|T_{\mathbf{e},w_1}(F)\|_2 \text{ s.t. } \|T_{\mathbf{e},w_2}(F)\|_\infty \leq \mathbf{g} \right\}$$

*The solution to this problem is non trivial if  $\mathbf{g} < \|T_{\mathbf{e},w_2}(\text{Kalman})\|_\infty$ .*

*Up to now there not exist a closed-form solution. It can be proven that the optimal mixed solution  $F^o$  is on the "boundary", i.e. such that*

$$\|T_{\mathbf{e},w_2}(F^o)\|_\infty = \mathbf{g}$$

---

Obviously, the mixed problem makes sense since there exist an infinite number of filters  $F$  (a parametrized family) such that  $\|T_{\mathbf{e},w_2}(F)\|_\infty \leq \mathbf{g}$

## OBSERVATIONS

The gain  $F_3$  of the predictor ensuring the norm less than  $\mathbf{g}$  can be parametrized in the following way:  $F_3 = W_1^{-1} W_2$ , where  $(W_1, W_2)$  satisfies an elliptic equation.

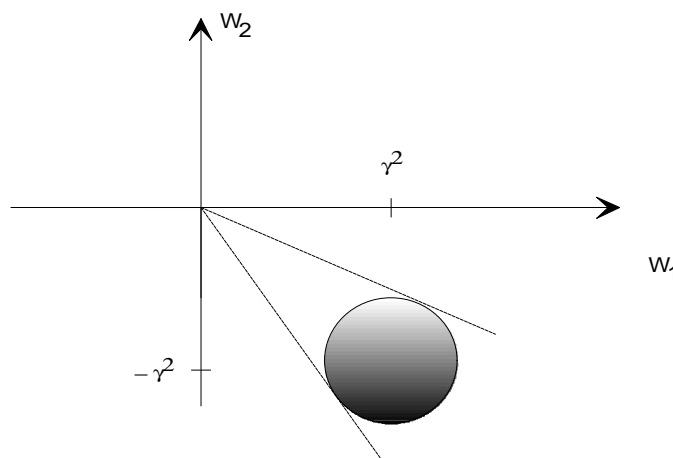
### Convex Optimization

$$\min \text{tr}(XW) \geq \min \left\| T_{ew}(W_1^{-1} W_2) \right\|_2^2, \quad W = \begin{bmatrix} W_1 & W_2 \\ W_2' & W_3 \end{bmatrix}$$

where  $X$  depends from the data. If  $w_1 = w_2$ , then one obtains the central predictor. In this way one simply minimizes an upper bound of the real cost. The solution may be too conservative.

### Search for boundary values

The boundary values are (norm equal to  $\mathbf{g}$ ) on the tangent sets between the ellipsoid and the planes passing through the origin.

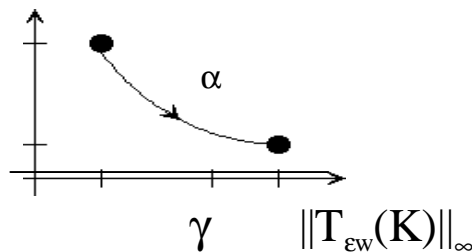


## a - PROCEDURE

We can parametrize with a scalar variable a family of predictor (defined by the only gain  $K$ ) which ensure the norm less than  $\mathbf{g}$  with minimal two norm.

### Idea

$$\|T_{\varepsilon w}(K)\|_2$$



$$K(\alpha) = (1-\alpha)K_c + \alpha[AQ(\alpha)C' + BD'] (CQ(\alpha)C' + DD')^{-1}$$

$$Q = \bar{A}Q\bar{A}' + \bar{B}\bar{B}' - (2\mathbf{a} - \mathbf{a}^2) (\bar{A}QC' + \bar{B}D') (CQC' + DD')^{-1} (\bar{A}QC' + \bar{B}D')$$

$$\bar{A} = A - K_c C, \quad \bar{B} = B - K_c D$$

where  $K_c$  is such that  $\|T_{\varepsilon w}(K_c)\|_{\infty} < \mathbf{g}$

### Theorem 8

Assume that  $(A, C)$  detectable and  $(A, B, C, D)$  without unstable zeros. Then there always exists  $Q(\mathbf{a}) \geq 0$ , " $\mathbf{a} \in \hat{\mathbf{I}} [0, 2]$ " with  $A - K(\mathbf{a})C$  stable. The norm  $\|T_{\varepsilon w}(K(\alpha))\|_2$  is nondecreasing, " $\mathbf{a} \in \hat{\mathbf{I}} [0, 1]$ ". There exists  $\mathbf{a}^* \in \hat{\mathbf{I}} [0, 2]$  such that  $\|T_{\varepsilon w}(K(\alpha^*))\|_{\infty} = \mathbf{g}$  and  $\|T_{\varepsilon w}(K(\alpha))\|_2$  is minimized.

---

We need to compute the value of  $\alpha$  closer to one and such that  $\|T_{\varepsilon w}(K(\alpha^*))\|_{\infty} = \mathbf{g}$  (search)



## FINITE HORIZON

$$S : \begin{cases} x(t+1) = Ax(t) + Bw(t) \\ z(t) = Lx(t) \\ y(t) = Cx(t) + Dw(t) \end{cases}$$

**Finite horizon problem in  $[1, T]$ :**

Find a linear causal filter based on  $y(1), y(2), \dots, y(t), t \leq T$  providing an estimate  $\hat{z}(t)$  guaranteeing a level of attenuation  $\mathbf{g} > 0$ , i.e. such that  $J_F < 0$  for each  $[w(\cdot) \ (x(0) - \hat{x}_0)'] \neq 0$ , where

$$J_F = \sum_{t=1}^T \|z(t) - \hat{z}(t)\|^2 - \mathbf{g}^2 \left[ \sum_{t=0}^T \|w(t)\|^2 + (x(0) - \hat{x}_0)' \Psi (x(0) - \hat{x}_0) \right]$$

**Simplifying assumptions:**

$$DB' = 0, \quad DD' = I$$

**Theorem 9**

A filter guaranteeing an attenuation level  $\gamma$  exists iff there exist two positive definite matrices  $Q, S$  such that

$$Q(t+1) = AS(t)^{-1}A + BB'$$

$$S(t+1) = Q(t+1)^{-1} + C'C - L'L / \mathbf{g}^2, \quad S(0) = \Psi'$$

---

The vector  $\hat{x}_0$  is an a-priori estimate of  $x_0$ , and  $R > 0$  is a weighting matrix (it plays the role played by the initial covariance matrix in the Kalman filter design). Putting the two equations together we get the difference

Riccati equation:  $Q(t+1) = A \left[ Q(t)^{-1} + C'C - L'L / \mathbf{g}^2 \right]^{-1} + BB'$ , with  $Q(1) = A\Psi^{-1}A' + BB'$ .

## SOLUTION OF THE FINITE HORIZON PROBLEM

$$\mathbf{h}(t+1) = A\mathbf{h}(t) + K(t+1)[y(t+1) - CA\mathbf{h}(t)]$$

$$\hat{\mathbf{z}}(t) = L\mathbf{h}(t)$$

*The gain is given by:*

$$K(t) = Q(t)C'[I + CQ(t)C']^{-1}$$

*where  $Q(t)$  is the solution of the difference Riccati equation*

### QUESTIONS

*Under which conditions it is possible to guarantee the existence of the time-varying filter for an arbitrarily long interval?*

*Under which conditions it is possible to guarantee the convergence for  $t \rightarrow \infty$  to the H $\infty$  filter?*

---

The time-varying filter can be written as

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + AK(t)[y(t) - C\mathbf{x}(t)]$$

$$\hat{\mathbf{z}}(t) = L\mathbf{x}(t) + LK(t)[y(t) - C\mathbf{x}(t)]$$

The gains  $F_1(t) = AK(t)$ ,  $F_2(t) = LK(t)$  have the same structure as in the stationary case (uncorrelated noises).

## FEASIBILITY AND CONVERGENCE

1)  $A$  invertible

2) There exists the stabilizing solution  $Q_s$  of

$$Q = A \left[ Q^{-1} + \tilde{C}' R^{-1} \tilde{C} \right]^{-1} A' + BB'$$

3) The pair  $(A, B)$  is reachable

Thanks to these assumptions, there exists the unique (positive semidefinite) solution of the Lyapunov equation

$$Y = \left[ Q_s^{-1} + \tilde{C}' R^{-1} \tilde{C} \right]^{-1} A' Y A \left[ Q_s^{-1} + \tilde{C}' R^{-1} \tilde{C} \right]^{-1} - \left[ \tilde{C}' (R + \tilde{C} Q_s \tilde{C}')^{-1} \tilde{C} \right]$$

### Theorem 10

(i) The solution  $Q(t)$  of the difference Riccati equation is feasible, " $t > 0$ , if, for some  $\mathbf{e}$  :

$$0 < Q(1) < Q_s + \left\{ \left[ Y - \left[ \tilde{C}' (R + \tilde{C} Q_s \tilde{C}')^{-1} \tilde{C} \right] \right] + \mathbf{e} I \right\}^{-1}$$

(ii) The solution  $Q(t)$  of the difference Riccati equation is feasible and converges to the stabilizing solution  $Q_s$  if

$$0 < Q(1) < Q_s + [Y + \mathbf{e} I]^{-1}$$

---

It is possible to see that the conditions introduced in the theorem are also necessary in two significant cases:  $L=0$  ( $\gamma \rightarrow \infty$ ) and  $C=0$ . In the first case  $Y=0$  and therefore one recovers the condition of convergence for the Kalman filter ( $Q(1) > 0$ ), whereas, in the second case the condition of convergence is equivalent to  $S(1) = Q(1)^{-1} + C'C - LL\gamma^2 > S_a$ , where  $S_a$  is the antistabilizing solution of  $S = (AS^{-1}A' + BB')^{-1} + C'C - LL\gamma^2$

## g-SWITCHING

If  $Q(1)$  is "too large", nothing can be said on the convergence of  $Q(t, \gamma_s)$  towards the stabilizing solution  $Q_s(\gamma)$  associated with a given value of  $\gamma = \gamma_s$ .

The convergence condition is:  $0 < Q(1) < Q_s(\mathbf{g}) + [Y(\mathbf{g}) + \mathbf{e}I]^{-1}$ .

### Theorem 11

- (i) The solution  $Y(\gamma)$  of the Lyapunov equation goes to zero for  $\gamma \rightarrow \infty$
- (ii) The stabilizing solution  $Q_s(\gamma)$  of the Riccati equation increases as  $\gamma$  increases

Thanks to point (i) of the theorem, if  $Q(1)$  does not satisfy the convergence condition, one can find a value of  $\gamma = \gamma_1 > \gamma_s$  such that the convergence condition is satisfied for  $\gamma = \gamma_1$ . The solution of the Riccati equation with  $\gamma = \gamma_1$  converges to  $Q_s(\gamma_1)$ . On the other hand, thanks to point (ii) of the theorem, one can say that

$$Q_s(\gamma_1) \leq Q_s(\gamma_s) < Q_s(\gamma_s) + (Y(\gamma_s)^{-1} + \mathbf{e}I)^{-1}$$

So that  $\mathbf{e} > 0$  and  $t_e > 0$  are such that

$$Q(t_e, \gamma_1) < Q_s(\gamma_s) + (Y(\gamma_s)^{-1} + \mathbf{e}I)^{-1}.$$

Finally, imposing

$$\mathbf{g}(t) = \begin{cases} \mathbf{g}_1, & 1 \leq t < t_e \\ \mathbf{g}_s, & t \geq t_e \end{cases}$$

the solution  $Q(t, \gamma(t))$  tends to  $Q_s(\gamma_s)$  as  $t \rightarrow \infty$ .

The  $\gamma$ -switching strategy modifies the cost:

$$\tilde{J}_F = \sum_{i=1}^T \frac{\|r(t) - \hat{r}(t)\|^2}{\mathbf{g}^2(t)} - \left[ \sum_{t=0}^T \|w(t)\|^2 + (x(0) - \hat{x}_0)' \Psi (x(0) - \hat{x}_0) \right]$$

## EXAMPLE

$$A=0.5, B=[1 \ 0], C=3, D=[0 \ 1], L=2, T=1/12$$

$$Q(t+1) = 0.25 \frac{1}{Q(t)^{-1} + 9 - 4g^{-2}} + 1, \quad Q(1) = 4$$

*Feasibility condition*

$$Q(t) < \frac{g^2}{4 - 9g^2}$$

*Algebraic Riccati equation*

$$Q = 0.25 \frac{1}{Q^{-1} + 9 - 4g^{-2}} + 1 \quad \textcircled{R} \quad \text{there existse } Q_s(\gamma) \text{ for } \gamma > 0.6576$$

*Choice:  $\gamma_s = 0.66$*

*Hence,  $Q_s(\gamma_s) = 1.5318$  is the stabilizing solution*

*$Q(1) < 5.4724$  feasibility condition*

*$Q(1) < 3.572$  convergence condition*

*Integrating the equation with  $\gamma = \gamma_s$  it results  $Q(t) = 4, 4.7167, 9.5394, -2.2089, 0.6066, \dots$  and then the feasibility is lost after three steps. Taking  $\gamma_1$  in such a way that  $4 < Q_s(\gamma_s) + Y^{-1}(\gamma_1)$ . For example  $\gamma_1$  close to  $\gamma_s$  satisfying this inequality is  $\gamma_1 = 0.661$ . Integrating the equation with  $\gamma = \gamma_1$  it results  $Q(t) = 4, 3.63, 3.07, \dots$  Since  $Q(3) < 3.572$ , it is possible to impose  $\gamma = \gamma_s$  in the equation at time  $t=3$  so that the solution converges to  $Q_s(\gamma_s)$ .*

## REFERENCES

- T.Basar, P.Bernhard, "H<sub>∞</sub> -Optimal Control and related minimax design problems- a dynamic game approach", Birkhauser, 1991.
- P.Bolzern, P.Colaneri, G.De Nicolao, "Transient and asymptotic analysis of discrete-time H<sub>∞</sub> filters", *European Journal of Control*, to appear.
- J.C.Geromel, P.L.D.Peres, S.R.Souza, "A convex approach to the mixed H<sub>2</sub>/H<sub>∞</sub> control problem for discrete-time uncertain systems", *SIAM J. Control and Optimization*, Vol. 33, 6, pp. 1816-1833, 1995.
- M.Green, K.Glover, D.Limebeer, J.Doyle, "A J-spectral factorization approach to H<sub>∞</sub> control", *SIAM J. Control and Optimization*, 25,6, pp. 1350-1371.
- M.J.Grimble, A. El Sayed, "Solution of the H<sub>∞</sub> optimal linear filtering problem for discrete-time systems", *IEEE Trans. ASSP*, 38, 7, pp. 1092-1104, 1990.
- W.Haddad, D.Bernstein, D.Mustafa, "Mixed H<sub>2</sub>/H<sub>∞</sub> - regulation and estimation", *Systems and Control Letters*, 16, 1991.
- B.Hassibi, T.Kailath, "H<sub>∞</sub> bounds for recursive-least square algorithms", *33rd Conference on Decision and Control*, Lake Buena Vista, pp. 3927-3938, 1994.
- B.Hassibi, T.Kailath, "H<sub>∞</sub> optimality of the LMS algorithms", *IEEE Trans. on Automatic Control*, 44, 2, pp. 267-280, 1996.
- B.Hassibi, A.H.Sayed, T.Kailath, "Linear estimation in Krein spaces - Part II: Applications", *IEEE Trans. AC*, 41, pp. 34-49, 1996.
- B.Hassibi, A.H.Sayed, T.Kailath, "Linear estimation in Krein spaces - Part I: Theory", *IEEE Trans. AC*, 41, pp. 18-33, 1996.
- B.Hassibi, A.H.Sayed, T.Kailath, "H<sub>∞</sub> Optimality of the LMS algorithm", *IEEE Trans. ASSP*, 44, pp. 267-1022, 1996.
- D.H.Jacobson, "Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic games", *IEEE Trans. Automatic Control*, AC-18, 2, 1973.
- P.Lancaster, L.Rodman, "Algebraic Riccati Equations", Oxford Series Pub., 1995.
- D.J.Limebeer, M.Green, D.Walker, "Discrete-time H<sub>∞</sub> control", *Proc. 28th Conference on Decision and Control*, Tampa (USA), pp. 392-396, 1989.

- H.Rotstein, M.Sznaier, M.Idan, " $H_2/H_\infty$  filtering theory and an aerospace application", *Int. J. Robust and Nonlinear Control*, 6, pp. 347-366, 1996.
- J.Speyer, J.Deyst, D.H.Jacobson, "Optimization of stochastic linear systems with additive measurement and process noise using exponential performance criteria", *IEEE Trans. Automatic Control*, AC-19, pp.358-366, 1974.
- J.L.Speyer, C.Fan, R.N.Banavar, "Optimal stochastic estimation with exponential cost criteria", *Proc. Conference on Decision and Control*, pp. 2293-2298, 1992.
- A.A.Stoorvogel, A.Saberi, B.M.Chen, "The discrete-time  $H_\infty$  control problem with measurement feedback", *Int. J. Robust and Nonlinear Contr.*, 4, pp. 457-479, 1994.
- Y.Theodor, U.Shaked, N.Berman, "Time-domain  $H_\infty$  identification", *IEEE Trans. on Aut. Control*, 41, pp. 1019-1022, 1996.
- M.Vidyasagar, "Control System Synthesis: a factorization approach", MIT Press Series, 1985.
- P.Whittle, "Risk Sensitive Optimal Control", New York: Wiley, 1990.
- I.Yaesh, U.Shaked, "Game theory approach to optimal linear estimation in the minimum  $H_\infty$  norm sense", *28th Conference on Decision and Control*, Tampa, pp. 421-425, 1989.
- I.Yaesh, U.Shaked, "A transfer function approach to the problem of discrete-time systems:  $H_\infty$ -optimal linear control and filtering", *IEEE Trans. on Automatic Control*, 36, 11, pp. 1264-1271, 1991.
- I.Yaesh, U.Shaked, "Game theory approach to state-estimation of linear discrete-time processes and its relation to  $H_\infty$ -optimal estimation", *Int. J. on Control*, 55, 6, pp. 1443-1452, 1992.